A connectedness theorem over the spectrum of a formal power series ring

AUTHOR(S):

Kawakita, Masayuki

A CONNECTEDNESS THEOREM OVER THE SPECTRUM OF A FORMAL POWER SERIES RING

MASAYUKI KAWAKITA

ABSTRACT. We study the connectedness of the non-klt locus over the spectrum of a formal power series ring. In dimension 3, we prove the existence and normality of the smallest lc centre, and apply it to the ACC for minimal log discrepancies greater than 1 on smooth 3-folds.

1. INTRODUCTION

The vanishing theorem by Kodaira [18] is one of the most basic tools in algebraic geometry in characteristic zero. It is reasonable to expect a vanishing theorem on excellent schemes, but it is annoyingly unknown besides the work on surfaces by Lipman [22]. Precisely, we are interested in the relative Kodaira vanishing for a birational morphism over the spectrum of a formal power series ring $R = K[[x_1,\dots,x_d]]$ for a field K of characteristic zero. We mean by an R-variety an integral separated scheme of finite type over $\operatorname{Spec} R$.

Conjecture 1.1. *Let $f\colon Y \to X$ be a projective birational morphism of regular R-varieties and L an f-ample divisor on Y. Then $R^i f_*\mathscr{O}_Y(K_{Y/X}+L) = 0$ for $i \ge 1$. Here the relative canonical divisor $K_{Y/X}$ is defined by the 0-th Fitting ideal of $\Omega_{Y/X}$.*

We shall not deal with this algebraic conjecture. Instead, we study the *connectedness lemma* by Shokurov [29] and Kollár [19], which is an important geometric application of the vanishing theorem in birational geometry. It claims for a proper morphism $f\colon Y \to X$, the fibrewise connectedness of the non-klt locus of a subpair (Y,Δ) such that Δ is effective outside a locus in X of codimension at least 2 and such that $-(K_Y+\Delta)$ is f-nef and f-big. We shall verify it for a germ at a regular point of X in the case when f is isomorphic outside the central fibre (Theorem 3.1). Investigating further in dimension 3, we obtain a desirable result on the smallest lc centre of a pair on a regular R-variety of dimension 3.

Theorem 1.2. *Let $P \in (X,\mathfrak{a})$ be a germ of an lc but not klt pair of a regular R-variety X of Krull dimension 3 and an $\mathbb{R}$-ideal $\mathfrak{a}$ on X. Then the smallest lc centre of $(X,\mathfrak{a})$ exists and it is normal.*

We reduce to the case $X = \operatorname{Spec} R$ with K an algebraically closed field k. Theorem 3.1, the fibrewise connectedness, is proved by approximating the effective $\mathbb{R}$-divisor $f_*\Delta$ by an $\mathfrak{m}$-primary $\mathbb{R}$-ideal $\mathfrak{a}\langle l\rangle$, where $\mathfrak{m}$ is the maximal ideal sheaf, such that the non-klt locus of the subtriple coming from $\mathfrak{a}\langle l\rangle$ coincides with the central fibre of the original non-klt locus. The $\mathfrak{a}\langle l\rangle$ is descended to $\mathbb{A}^d_k = \operatorname{Spec} k[x_1,\dots,x_d]$, on which the connectedness lemma is applied. The existence of the smallest lc centre in Theorem 1.2 is a corollary to Theorem 3.1. The hardest part of Theorem 1.2 is the normality of the smallest lc centre C which is a curve. We construct an ideal

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sheaf $\mathfrak{n}_a$ on the normalisation C_Y of C with $f_C\colon C_Y \to C$ which satisfies $f_{C*}\mathfrak{n}_a \subset \mathscr{O}_C$ and $\mathscr{O}_C/f_{C*}\mathfrak{n}_a \simeq f_{C*}\mathscr{O}_{C_Y}/f_{C*}\mathfrak{n}_a$. Then we obtain the isomorphism $\mathscr{O}_C \simeq f_{C*}\mathscr{O}_{C_Y}$ meaning the normality of C.

Our motivation for excellent schemes stems from the notion of a generic limit of ideals due to de Fernex and Mustaţă [8]. The generic limit was used to prove the ascending chain condition (ACC) for log canonical thresholds on smooth varieties [7], the approach of which works even for the study of minimal log discrepancies [16]. We shall apply Theorem 1.2 to the ACC conjecture for minimal log discrepancies by Shokurov [28], [30] and Cascini, McKernan [24] in the case of smooth 3-folds, and settle the part of minimal log discrepancies greater than 1.

Theorem 1.3. *Fix subsets $I \subset (0,\infty)$ and $J \subset (1,3]$ both of which satisfy the descending chain condition. Then there exist finite subsets $I_0 \subset I$ and $J_0 \subset J$ such that if $P \in (X, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$ is a germ of a pair of a smooth variety X of dimension 3 and an $\mathbb{R}$-ideal $\mathfrak{a}$ on X with all $\mathfrak{a}_j$ non-trivial at P, all $r_j \in I$ and $\mathrm{mld}_P(X,\mathfrak{a}) \in J$, then all $r_j \in I_0$ and $\mathrm{mld}_P(X,\mathfrak{a}) \in J_0$.*

The generic limit a of $\mathbb{R}$-ideals $\mathfrak{a}_i$ on $P \in X = \operatorname{Spec} k[[x_1,\dots,x_d]]$ is an $\mathbb{R}$-ideal on $P_K \in X_K = \operatorname{Spec} K[[x_1,\dots,x_d]]$ with a field extension K of k. The ACC for minimal log discrepancies on smooth d-folds is reduced to the stability $\mathrm{mld}_{P_K}(X_K,\mathsf{a}) = \mathrm{mld}_P(X,\mathfrak{a}_i)$ for general i. We prove it when (X_K,a) is a klt pair, or even a plt pair whose lc centre has an isolated singularity, by our previous arguments [14], [15]. In dimension 3, only the case when (X_K,a) has the smallest lc centre of dimension 1 remains. In this case, the estimate $\mathrm{mld}_{P_K}(X_K,\mathsf{a}) \le 1$ is derived from Theorem 1.2, which is enough to prove Theorem 1.3.

The structure of the paper is as follows. After reviewing the basics of singularities in Section 2, we study the connectedness of the non-klt locus and establish Theorem 1.2 in Section 3. We discuss the ACC for minimal log discrepancies from the point of view of generic limits in Section 4. The stability of minimal log discrepancies in the klt and plt cases is shown in Section 5. Theorem 1.3 is completed in Section 6. The appendix exposing generic limits is attached.

Throughout this paper, k is an algebraically closed field of characteristic zero.

Remark 1.4. Recently, Chatzistamatiou and Rülling proved that the higher direct images of the structure sheaf vanish for a projective birational morphism of regular excellent schemes [4].

2. Singularities

We review the basics of singularities in birational geometry. A good reference is [21]. A *variety* is an integral separated scheme of finite type over $\operatorname{Spec} k$. A *germ* of a scheme is considered at a closed point. The *dimension* of a scheme means the Krull dimension.

An $\mathbb{R}$-*ideal* on a noetherian scheme X is a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ on X with positive real exponents r_j. The $\mathfrak{a}$ to the *power* of $t > 0$ is $\mathfrak{a}^t := \prod_j \mathfrak{a}_j^{tr_j}$. The *co-support* $\operatorname{Cosupp}\mathfrak{a}$ of $\mathfrak{a}$ is the union of all $\operatorname{Supp}\mathscr{O}_X/\mathfrak{a}_j$. The *pull-back* of $\mathfrak{a}$ by a morphism $Y \to X$ is $\mathfrak{a}\mathscr{O}_Y := \prod_j(\mathfrak{a}_j\mathscr{O}_Y)^{r_j}$. The $\mathbb{R}$-ideal $\mathfrak{a}$ is said to be *invertible* if all $\mathfrak{a}_j$ are invertible. In this case, if in addition X is normal, then the $\mathbb{R}$-divisor $A = \sum_j r_jA_j$ with $\mathfrak{a}_j = \mathscr{O}_X(-A_j)$ is called the $\mathbb{R}$-divisor *defined by* $\mathfrak{a}$.

Let Z be an irreducible closed subset of X. We write η_Z for the generic point of Z. The *order* of $\mathfrak{a}$ along Z is $\mathrm{ord}_Z \mathfrak{a} = \sum_j r_j \mathrm{ord}_Z \mathfrak{a}_j$, where $\mathrm{ord}_Z \mathfrak{a}_j$ is the maximal $\nu \in \mathbb{N} \cup \{+\infty\}$ satisfying $\mathfrak{a}_j \mathscr{O}_{X,\eta_Z} \subset \mathscr{I}_Z^{\nu} \mathscr{O}_{X,\eta_Z}$ for the ideal sheaf $\mathscr{I}_Z$ of Z.

We treat a *triple* $(X, \Delta, \mathfrak{a})$ which consists of a normal variety X, an effective $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and an $\mathbb{R}$-ideal $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ on X. A prime divisor E on a normal variety Y with a birational morphism $f\colon Y \to X$ is called a divisor *over* X, and the closure $\overline{f(E)}$ of the image on X is called the *centre* of E on X and denoted by $c_X(E)$. We denote by $\mathscr{D}_X$ the set of all divisors over X. The *log discrepancy* of E with respect to $(X, \Delta, \mathfrak{a})$ is

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \mathrm{ord}_E K_{Y/(X,\Delta)} - \mathrm{ord}_E \mathfrak{a},$$

where $K_{Y/(X,\Delta)} := K_Y - f^*(K_X + \Delta)$ and $\mathrm{ord}_E \mathfrak{a} := \mathrm{ord}_E \mathfrak{a}\mathscr{O}_Y$. Note that $c_X(E)$ and $a_E(X, \Delta, \mathfrak{a})$ are determined by the valuation on the function field of X given by E.

For an irreducible closed subset Z of X, the *minimal log discrepancy* of $(X, \Delta, \mathfrak{a})$ at η_Z is

$$\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) := \inf\{a_E(X, \Delta, \mathfrak{a}) \mid E \in \mathscr{D}_X,\ c_X(E) = Z\}.$$

It is either a non-negative real number or $-\infty$. We say that $E \in \mathscr{D}_X$ *computes* $\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ if $c_X(E) = Z$ and $a_E(X, \Delta, \mathfrak{a}) = \mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$ (or is negative when $\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = -\infty$). We often reduce to the case when Z is a closed point by the relation $\mathrm{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = \mathrm{mld}_P(X, \Delta, \mathfrak{a}) - \dim Z$ for a general closed point $P \in Z$ (cf. [3, Proposition 2.1]).

The triple $(X, \Delta, \mathfrak{a})$ is said to be *log canonical* (*lc*) (resp. *Kawamata log terminal* (*klt*)) if $a_E(X, \Delta, \mathfrak{a}) \ge 0$ (resp. > 0) for all $E \in \mathscr{D}_X$. It is said to be *purely log terminal* (*plt*) (resp. *canonical*, *terminal*) if $a_E(X, \Delta, \mathfrak{a}) > 0$ (resp. ≥ 1, > 1) for all exceptional $E \in \mathscr{D}_X$. The log canonicity of $(X, \Delta, \mathfrak{a})$ about $P \in X$ is equivalent to $\mathrm{mld}_P(X, \Delta, \mathfrak{a}) \ge 0$. Let Y be a normal variety with a birational morphism to X. A centre $c_Y(E)$ with $a_E(X, \Delta, \mathfrak{a}) \le 0$ is called a *non-klt centre* on Y of $(X, \Delta, \mathfrak{a})$. The union of all non-klt centres on Y is called the *non-klt locus* on Y and denoted by $\mathrm{Nklt}_Y(X, \Delta, \mathfrak{a})$. When we say just a non-klt centre or the non-klt locus, we mean that it is on X.

A *log resolution* of $(X, \Delta, \mathfrak{a})$ is a projective morphism $f\colon Y \to X$ from a regular variety Y such that

(i) $\mathrm{Exc} f$ is a divisor and $\mathfrak{a}\mathscr{O}_Y$ is invertible,
(ii) $\mathrm{Exc} f \cup \mathrm{Supp}\,\Delta_Y \cup \mathrm{Cosupp}\,\mathfrak{a}\mathscr{O}_Y$ is a simple normal crossing (snc) divisor, where Δ_Y is the strict transform of Δ, and
(iii) f is isomorphic on the locus U in X with U regular, $\mathfrak{a}|_U$ invertible and $\mathrm{Supp}\,\Delta|_U \cup \mathrm{Cosupp}\,\mathfrak{a}|_U$ snc.

A *stratum* (resp. an *open stratum*) of an snc divisor $\sum_{i \in I} E_i$ is an irreducible component of $\bigcap_{i \in J} E_i$ (resp. $\bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i$) for a subset J of I.

By allowing a not necessarily effective $\mathbb{R}$-divisor Δ, one can consider a *subtriple* $(X, \Delta, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$. The notions of log discrepancies and lc/klt singularities are extended for subtriples. Let $f\colon Y \to X$ be a birational morphism from a regular variety Y such that $\mathrm{Exc} f$ is a divisor $\sum_i E_i$. The *weak transform* on Y of $\mathfrak{a}$ is the $\mathbb{R}$-ideal $\mathfrak{a}_Y = \prod_j \mathfrak{a}_{jY}^{r_j}$ with $\mathfrak{a}_{jY} = \mathfrak{a}_j \mathscr{O}_Y(\sum_i (\mathrm{ord}_{E_i} \mathfrak{a}_j) E_i)$.

Definition 2.1. Notation as above. The *pull-back* of $(X, \Delta, \mathfrak{a})$ by f is the subtriple $(Y, \Delta_Y, \mathfrak{a}_Y)$ where $\Delta_Y = -K_{Y/(X,\Delta)} + \sum_{ij} (r_j \mathrm{ord}_{E_i} \mathfrak{a}_j) E_i$.

We have $a_E(X,\Delta,\mathfrak{a}) = a_E(Y,\Delta_Y,\mathfrak{a}_Y)$ for any $E \in \mathscr{D}_Y$. In particular when f is proper, $(X,\Delta,\mathfrak{a})$ is lc (resp. klt) if and only if so is $(Y,\Delta_Y,\mathfrak{a}_Y)$. We use the notation $\mathrm{Nklt}_Y(X,\Delta,\mathfrak{a})$ also for a subtriple $(X,\Delta,\mathfrak{a})$.

These definitions are extended on schemes over a field K of characteristic zero and even over a formal power series ring $R = K[[x_1,\dots,x_d]]$ by the existence of log resolutions due to Hironaka [12] and Temkin [32], [33]. This extension is studied by de Fernex, Ein and Mustaţă [7], [8]. We mean by an *R-variety* an integral separated scheme of finite type over $\operatorname{Spec} R$.

The canonical divisor K_X on a normal R-variety X is defined by the isomorphism $\mathscr{O}_X(K_X)|_U \simeq \bigwedge^r \Omega'_{X/K}|_U$ on the regular locus U of X, where $\Omega'_{X/K}$ is the *sheaf of special differentials* in [7] and r is its rank. The relative canonical divisor is well understood for a birational morphism of regular R-varieties.

Lemma 2.2 ([7, Remark A.12])**.** *Let $Y \to X$ be a proper birational morphism of regular R-varieties. Then $K_{Y/X}$ is the effective divisor defined by the* 0*-th Fitting ideal of $\Omega_{Y/X}$. In particular, $K_{Y/X}$ is independent of the structure of X as an R-variety.*

The log discrepancies are preserved by field extensions and completions.

Corollary 2.3. *Let $Y \to X$ be as in Lemma* 2.2*. Take an R'-variety X' as in* (i)*,* (ii) *or* (iii) *below and set a morphism $Y' = Y \times_X X' \to X'$ of R'-varieties.*

(i) *X' is a component of $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ with $R' = \widehat{R \otimes_K K'}$ for a field extension K' of K.*

(ii) *$X' = \operatorname{Spec} \widehat{\mathscr{O}_{X,P}}$ for a germ $P \in X$, which admits the structure of an R'-variety for a suitable $R' = K'[[x_1,\dots,x_{d'}]]$ by Cohen's structure theorem* [5]*.*

(iii) *$X' = X$ with another structure morphism $X \to \operatorname{Spec} R'$.*

Then $K_{Y'/X'}$ is the pull-back of $K_{Y/X}$. In particular, for an $\mathbb{R}$-ideal $\mathfrak{a}$ on X, a divisor E over X and a germ $P \in X$, one has $a_{E'}(X',\mathfrak{a}\mathscr{O}_{X'}) = a_E(X,\mathfrak{a})$ for a component E' of $E \times_X X'$ and $\mathrm{mld}_{P'}(X',\mathfrak{a}\mathscr{O}_{X'}) = \mathrm{mld}_P(X,\mathfrak{a})$ for a point P' of $P \times_X X'$.

This is by the regularity of the morphism $X' \to X$. The cases (i) and (ii) for $R = K$ are stated in [7, Lemma 2.14, Propositions 2.11, A.14] even for a normal ($\mathbb{Q}$-Gorenstein) K-variety X.

Suppose that $(X,\Delta,\mathfrak{a})$ is an lc triple. Then a non-klt centre (on X) of $(X,\Delta,\mathfrak{a})$ is often called an *lc centre*. An lc centre which is minimal with respect to inclusions is called a *minimal lc centre*. When we work over a germ $P \in X$, the following definition makes sense.

Definition 2.4. Let $P \in (X,\Delta,\mathfrak{a})$ be a germ of an lc triple. The *smallest lc centre* is an lc centre of $(X,\Delta,\mathfrak{a})$ contained in every lc centre.

If X is a variety, then the smallest lc centre exists and it is normal [9, Theorem 9.1]. It is, however, unknown for R-varieties. Theorem 1.2 states that this is the case when X is a regular R-variety of dimension 3.

3. The smallest lc centre on a threefold

This section is devoted to the proof of Theorem 1.2. We work over a germ $P \in X$ of an R-variety with $R = K[[x_1,\dots,x_d]]$. The maximal ideal sheaf of $P \in X$ is denoted by $\mathfrak{m}$. When we discuss on the spectrum of a noetherian ring, we identify an ideal in the ring with its coherent ideal sheaf.

3.A. **A connectedness theorem.** We start with a connectedness theorem over X, Theorem 3.1. Though we impose the strong condition that f is isomorphic outside P, this theorem is sufficient in dimension 3 in order to derive the existence of the smallest lc centre, which will be seen in Subsection 3.C.

Theorem 3.1. *Let $P \in (X, \mathfrak{a})$ be a germ of a pair on a regular R-variety X and $f\colon Y \to X$ a proper birational morphism of regular R-varieties which is isomorphic outside P. Let Δ be an $\mathbb{R}$-divisor on Y with $f_*\Delta \ge 0$ such that $-(K_Y + \Delta)$ is f-nef. Then $\mathrm{Nklt}_Y(Y, \Delta, \mathfrak{a}\mathscr{O}_Y) \cap f^{-1}(P)$ is connected.*

We extract the case $\Delta = -K_{Y/X}$.

Corollary 3.2. *Let $P \in (X, \mathfrak{a})$ be a germ of a pair on a regular R-variety X and $f\colon Y \to X$ a proper birational morphism of regular R-varieties which is isomorphic outside P. Then $\mathrm{Nklt}_Y(X, \mathfrak{a}) \cap f^{-1}(P)$ is connected.*

The statement for $R = k$ is a special case of the connectedness lemma by Shokurov and Kollár [19, Theorem 17.4]. Their lemma can settle Theorem 3.1 in the case when $\mathfrak{a}$ is $\mathfrak{m}$-primary and Δ is f-exceptional. Write $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$.

Lemma 3.3. (i) *In order to prove Theorem* 3.1, *one may assume that $X = \operatorname{Spec} R$ with $K = k$, f is projective and Δ is f-exceptional.*
(ii) *Theorem* 3.1 *holds in the case when $X = \operatorname{Spec} R$ with $K = k$, f is projective, Δ is f-exceptional and all $\mathfrak{a}_j$ are $\mathfrak{m}$-primary ideals.*

Proof. (i) Take an isomorphism $\widehat{\mathscr{O}_{X,P}} \simeq K'[[x_1, \dots, x_{d'}]]$ with $K' = \mathscr{O}_{X,P}/\mathfrak{m}$ by Cohen's structure theorem and set $R' = k[[x_1, \dots, x_{d'}]]$ for the algebraic closure k of K'. Because the base change $\operatorname{Spec} R' \to X$ commutes with taking the non-klt locus by Corollary 2.3, we may assume $X = \operatorname{Spec} R$ with $K = k$ (the d may be changed). By the flattening theorem of Raynaud and Gruson [26, Théorème 1re 5.2.2], there exists a projective morphism $f'\colon Y' \to X$ from a regular R-variety Y' which is isomorphic outside P and factors through f. Replacing (Y, Δ) with its pull-back on Y', we may assume that f is projective. The $\Delta' := \Delta - f^*f_*\Delta$ is f-exceptional. Take an invertible $\mathbb{R}$-ideal $\mathfrak{d}$ on X which defines the $\mathbb{R}$-divisor $f_*\Delta \ge 0$. Then $\mathrm{Nklt}_Y(Y, \Delta, \mathfrak{a}\mathscr{O}_Y) = \mathrm{Nklt}_Y(Y, \Delta', \mathfrak{a}\mathfrak{d}\mathscr{O}_Y)$. Replacing Δ with Δ' and $\mathfrak{a}$ with $\mathfrak{a}\mathfrak{d}$, we may assume that Δ is f-exceptional.

(ii) We use the notation $\bar{R} = k[x_1, \dots, x_d]$ and $\mathbb{A}^d_k = \operatorname{Spec} \bar{R}$ with origin $\bar{P}$. By Proposition A.7, f is the base change of a projective morphism $\bar{f}\colon \bar{Y} \to \mathbb{A}^d_k$ and $\mathfrak{a}$ is the pull-back of the $\mathbb{R}$-ideal $\bar{\mathfrak{a}} = \prod_j (\mathfrak{a}_j \cap \bar{R})^{r_j}$. Then $f^{-1}(P) \simeq \bar{f}^{-1}(\bar{P})$ and Δ is the base change of an $\bar{f}$-exceptional $\mathbb{R}$-divisor $\bar{\Delta}$ such that $-(K_{\bar{Y}} + \bar{\Delta})$ is $\bar{f}$-nef. Thus $f^{-1}(P) \supset \mathrm{Nklt}_Y(Y, \Delta, \mathfrak{a}\mathscr{O}_Y) \simeq \mathrm{Nklt}_{\bar{Y}}(\bar{Y}, \bar{\Delta}, \bar{\mathfrak{a}}\mathscr{O}_{\bar{Y}})$, which is connected by [19, Theorem 17.4]. q.e.d.

For Theorem 3.1, we take a log resolution $q\colon W \to Y$ of $(Y, \Delta, \mathfrak{a}\mathfrak{m}\mathscr{O}_Y)$ and set the composition $g = f \circ q\colon W \to X$ as below.

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

We fix $\varepsilon > 0$ such that

$$F := \mathrm{Nklt}_W(Y, \Delta, \mathfrak{a}\mathscr{O}_Y) = \mathrm{Nklt}_W(Y, \Delta, \mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y). \tag{1}$$

We approximate $\mathfrak{a}^{1+\varepsilon}$ by an $\mathfrak{m}$-primary $\mathbb{R}$-ideal

$$\mathfrak{a}\langle l\rangle := \prod_j (\mathfrak{a}_j + \mathfrak{m}^l)^{r_j(1+\varepsilon)} \tag{2}$$

with $l \in \mathbb{N}$. We want to find a large l such that $\mathrm{Nklt}_Y(Y,\Delta,\mathfrak{a}\mathscr{O}_Y) \cap f^{-1}(P) = \mathrm{Nklt}_Y(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y)$ in order to apply Lemma 3.3(ii).

Since F and $g^{-1}(P)$ are divisors contained in an snc divisor, any irreducible component D of $F \cap g^{-1}(P)$ has $\mathrm{codim}_W D = 1$ or 2. Suppose $\mathrm{codim}_W D = 2$. Let $E_D \subset g^{-1}(P)$ and $F_D \subset F$ be the unique prime divisors such that $D \subset E_D \cap F_D$. We build a tower of blow-ups

$$\cdots \to W_i \xrightarrow{g_i} W_{i-1} \to \cdots \to W_0 = W \tag{3}$$

as follows. Set $W_0 := W$, $E_0 := E_D$ and $F_0 := F_D$. We construct inductively the blow-up $g_i \colon W_i \to W_{i-1}$ along D for $i = 1$ (resp. along $E_{i-1} \cap F_{i-1}$ for $i \geq 2$), and set E_i as the exceptional divisor of g_i, and F_i as the strict transform on W_i of F_D. The composition $g_1 \circ \cdots \circ g_i$ is denoted by $h_i \colon W_i \to W$.

Lemma 3.4. (i) $a_{E_i}(Y,\Delta,\mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y) \leq a_{E_D}(Y,\Delta,\mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y) - i\varepsilon\,\mathrm{ord}_{F_D}\mathfrak{a}$.
(ii) $h_{i*}\mathscr{O}_{W_i}(-aE_i) \subset \mathscr{O}_W(-aE_D) + \mathscr{O}_W(-F_D)$ *for any* $a \in \mathbb{N}$.

Proof. The (i) is just a computation using $a_{F_D}(Y,\Delta,\mathfrak{a}\mathscr{O}_Y) \leq 0$. The (ii) is from $h_{i*}\mathscr{O}_{W_i}(-aE_i) \cdot \mathscr{O}_{F_D} \subset h_{i*}\mathscr{O}_{F_i}(-aE_i|_{F_i}) = \mathscr{O}_{F_D}(-aE_D|_{F_D})$ via $F_i \simeq F_D$. q.e.d.

Lemma 3.5. *Suppose that* (Y,Δ) *is klt outside* $f^{-1}(P)$. *Then there exists* l *such that* $\mathrm{Nklt}_W(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y) = F \cap g^{-1}(P)$.

Proof. By (1), (2) and the assumption, $\mathrm{Nklt}_W(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y) \subset F \cap g^{-1}(P)$ for any l. Thus it suffices to prove that for every irreducible component D of $F \cap g^{-1}(P)$, there exists l_D such that D is a non-klt centre on W of $(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y)$ for any $l \geq l_D$. D has $\mathrm{codim}_W D = 1$ or 2. If $\mathrm{codim}_W D = 1$, then we may take any l_D such that $l_D\,\mathrm{ord}_D\mathfrak{m} \geq \mathrm{ord}_D\mathfrak{a}_j$ for all j. If $\mathrm{codim}_W D = 2$, then we take the tower of blow-ups in (3). Note $\mathrm{ord}_{F_D}\mathfrak{a} > 0$. By Lemma 3.4(i), we have $a_{E_i}(Y,\Delta,\mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y) \leq 0$ whenever $a_{E_D}(Y,\Delta,\mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y) \leq i\varepsilon\,\mathrm{ord}_{F_D}\mathfrak{a}$. Fix such i and take l_D such that $l_D\,\mathrm{ord}_{E_i}\mathfrak{m} \geq \mathrm{ord}_{E_i}\mathfrak{a}_j$ for all j. Then for $l \geq l_D$, $a_{E_i}(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y) = a_{E_i}(Y,\Delta,\mathfrak{a}^{1+\varepsilon}\mathscr{O}_Y) \leq 0$, so $D = c_W(E_i)$ is a non-klt centre on W of $(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y)$. q.e.d.

Proof of Theorem 3.1. After the reduction in Lemma 3.3(i), we take l in Lemma 3.5. Then

$$\begin{aligned}\mathrm{Nklt}_Y(Y,\Delta,\mathfrak{a}\mathscr{O}_Y) \cap f^{-1}(P) &= q(F \cap g^{-1}(P)) \\ &= q(\mathrm{Nklt}_W(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y)) = \mathrm{Nklt}_Y(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y).\end{aligned}$$

Apply Lemma 3.3(ii) to $(Y,\Delta,\mathfrak{a}\langle l\rangle\mathscr{O}_Y)$. q.e.d.

In our proof of Theorem 3.1, we do not know a relative vanishing for $g \colon W \to X$. Instead, consider a log resolution $f_l \colon Y_l \to X$ of $(X,\mathfrak{a}\langle l\rangle\mathfrak{m})$ which factors through f, and let $p_l \colon Y_l \to Y$ be the induced morphism as below. The l is not fixed here.

$$\begin{tikzcd} Y_l \arrow[r, "p_l"] \arrow[dr, "f_l"'] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

The f_l is isomorphic outside P. Let $(Y_l, \Delta_l, \mathscr{O}_{Y_l})$ be the pull-back of $(X, 0, \mathfrak{a}\langle l\rangle)$. Then we have a vanishing involving Δ_l, which will be used in Subsection 3.C.

Lemma 3.6. *Let $f_l = f \circ p_l \colon Y_l \to Y \to X$ be as above. Write $\lceil -\Delta_l \rceil = P_l - N_l$ by effective divisors P_l and N_l with no common divisors. Then*

$$R^1 f_*(p_{l*}\mathscr{O}_{Y_l}(-N_l)) = 0.$$

Proof. The sheaf $R^1 f_*(p_{l*}\mathscr{O}_{Y_l}(-N_l))$ is supported in P. Set $\widehat{\mathscr{O}_{X,P}} \simeq K'[[x_1, \dots, x_{d'}]]$ and $R' = k[[x_1, \dots, x_{d'}]]$ for the algebraic closure k of K', then R' is faithfully flat over $\mathscr{O}_{X,P}$. Hence taking the base change to $\operatorname{Spec} R'$, one can reduce to the case $X = \operatorname{Spec} R$ with $K = k$ by [10, Proposition III.1.4.15] and Corollary 2.3. By Proposition A.7, f_l is the base change of a projective morphism $\bar{f}_l \colon \bar{Y}_l \to \mathbb{A}^d_k$. The $\mathfrak{a}\langle l\rangle$ is the pull-back of an $\mathbb{R}$-ideal $\bar{\mathfrak{a}}\langle l\rangle$ on $\mathbb{A}^d_k$, and Δ_l is the base change of the $\mathbb{R}$-divisor $\bar{\Delta}_l$ on $\bar{Y}_l$ such that $(\bar{Y}_l, \bar{\Delta}_l, \mathscr{O}_{\bar{Y}_l})$ is the pull-back of $(\mathbb{A}^d_k, 0, \bar{\mathfrak{a}}\langle l\rangle)$.

Kawamata–Viehweg vanishing theorem [17], [35] implies $R^1 \bar{f}_{l*}\mathscr{O}_{\bar{Y}_l}(\lceil -\bar{\Delta}_l \rceil) = 0$. Since $X \to \mathbb{A}^d_k$ is flat, this is base-changed to $R^1 f_{l*}\mathscr{O}_{Y_l}(\lceil -\Delta_l \rceil) = 0$ by [10, Proposition III.1.4.15]. Thus, applying f_{l*} to the exact sequence

$$0 \to \mathscr{O}_{Y_l}(P_l - N_l) \to \mathscr{O}_{Y_l}(P_l) \to \mathscr{O}_{N_l}(P_l|_{N_l}) \to 0,$$

we obtain the surjection $\mathscr{O}_X = f_{l*}\mathscr{O}_{Y_l}(P_l) \twoheadrightarrow f_{l*}\mathscr{O}_{N_l}(P_l|_{N_l})$. This homomorphism is factored as $\mathscr{O}_X \to f_{l*}\mathscr{O}_{N_l} \hookrightarrow f_{l*}\mathscr{O}_{N_l}(P_l|_{N_l})$, so we have the surjection $\mathscr{O}_X \twoheadrightarrow f_{l*}\mathscr{O}_{N_l}$. Moreover, we have the base change $R^1 f_{l*}\mathscr{O}_{Y_l} = 0$ of the vanishing $R^1 \bar{f}_{l*}\mathscr{O}_{\bar{Y}_l} = 0$ [12, p.144 (2)]. Hence applying f_{l*} to the exact sequence

$$0 \to \mathscr{O}_{Y_l}(-N_l) \to \mathscr{O}_{Y_l} \to \mathscr{O}_{N_l} \to 0,$$

we obtain $R^1 f_{l*}\mathscr{O}_{Y_l}(-N_l) = 0$.

Leray spectral sequence $R^p f_*(R^q p_{l*}\mathscr{O}_{Y_l}(-N_l)) \Rightarrow R^{p+q} f_{l*}\mathscr{O}_{Y_l}(-N_l)$ gives an injection $R^1 f_*(p_{l*}\mathscr{O}_{Y_l}(-N_l)) \hookrightarrow R^1 f_{l*}\mathscr{O}_{Y_l}(-N_l)$, so $R^1 f_*(p_{l*}\mathscr{O}_{Y_l}(-N_l)) = 0$. q.e.d.

3.B. **Propositions in an arbitrary dimension.** We prepare two auxiliary propositions which can be stated independently of Theorem 1.2.

It is easy to see that a minimal lc centre of codimension 1 is normal.

Proposition 3.7. *Let $(X, \mathfrak{a})$ be a pair on a regular R-variety X, and S the union of all non-klt centres of codimension 1 of $(X, \mathfrak{a})$. Then every irreducible component of the non-normal locus of S is a non-klt centre of $(X, \mathfrak{a})$.*

Proof. S is Cohen–Macaulay since S is a Cartier divisor on a regular scheme X. Thus any irreducible component C of the non-normal locus of S has $\operatorname{codim}_X C = 2$ and $\operatorname{mult}_{\eta_C} S \geq 2$. Let E be the divisor over X obtained at η_C by the blow-up of X along C. Then $a_E(X, \mathfrak{a}) = 2 - \operatorname{ord}_E \mathfrak{a} \leq 2 - \operatorname{mult}_{\eta_C} S \leq 0$, so $C = c_X(E)$ is a non-klt centre of $(X, \mathfrak{a})$. q.e.d.

We can perturb $\mathfrak{a}$ to reduce to the case when every lc centre is minimal.

Proposition 3.8. *Let $(X, \mathfrak{a})$ be an lc pair on a klt R-variety X. Then there exists an $\mathbb{R}$-ideal $\mathfrak{a}'$ forming an lc pair $(X, \mathfrak{a}')$ such that a minimal lc centre of $(X, \mathfrak{a})$ is an lc centre of $(X, \mathfrak{a}')$ and vice versa.*

Proof. Let $\{Z_i\}_i$ be the set of all minimal lc centres of $(X, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$. For each Z_i, fix $E_i \in \mathscr{D}_X$ computing $\operatorname{mld}_{\eta_{Z_i}}(X, \mathfrak{a}) = 0$. Let $\mathscr{I}_Z$ be the ideal sheaf of $Z = \bigcup_i Z_i$, and take an integer l such that $l \operatorname{ord}_{E_i} \mathscr{I}_Z \geq \operatorname{ord}_{E_i} \mathfrak{a}_j$ for all i, j. Then $(X, \mathfrak{a}' :=$

$\prod_j(\mathfrak{a}_j+\mathscr{I}_Z^l)^{r_j})$ is lc, and Z_i is an lc centre of $(X,\mathfrak{a}')$ by $\mathrm{ord}_{E_i}\mathfrak{a}'=\mathrm{ord}_{E_i}\mathfrak{a}$. On the other hand, every lc centre of $(X,\mathfrak{a}')$ is an lc centre of $(X,\mathfrak{a})$ contained in $\mathrm{Cosupp}\,\mathfrak{a}'=Z$, so it equals some Z_i. q.e.d.

3.C. **The smallest lc centre on a threefold.** We proceed to the proof of Theorem 1.2. We may assume that P is not an lc centre of $(X,\mathfrak{a})$. By Proposition 3.8, we may assume that every lc centre of $(X,\mathfrak{a})$ is minimal.

The existence of the smallest lc centre is a consequence of Corollary 3.2.

Proof of the existence of the smallest lc centre. Let $\{Z_i\}_i$ be the set of all lc centres of $(X,\mathfrak{a})$, which are assumed to be minimal. Proposition 3.7 implies that $Z=\bigcup_i Z_i$ is regular outside P. Thus we have an embedded resolution $f\colon Y\to X$ of singularities of Z, in which f is isomorphic outside P and induces $f_Z\colon \bigsqcup_i Z_{iY}\to Z$ for the strict transform Z_{iY} of Z_i. By Corollary 3.2, $f_Z^{-1}(P)=\mathrm{Nklt}_Y(X,\mathfrak{a})\cap f^{-1}(P)$ is connected, that is, there exists only one lc centre of $(X,\mathfrak{a})$. q.e.d.

Remark 3.9. The above proof shows that if Z is the smallest lc centre of $(X,\mathfrak{a})$, then its normalisation $Z^\nu\to Z$ is a homeomorphism.

To complete Theorem 1.2, we must prove that the unique lc centre of $(X,\mathfrak{a})$ is normal. (Since we have assumed that every lc centre is minimal, the existence of the smallest lc centre means the uniqueness of lc centre.) If it is of dimension 2, then it is normal by Proposition 3.7. Thus, we may assume that $(X,\mathfrak{a})$ has the unique lc centre C which is of dimension 1.

We have an embedded resolution $f\colon Y\to X$ of singularities of C, in which f is isomorphic outside P and induces the normalisation $f_C\colon C_Y\to C$ for the strict transform C_Y of C. Note that $f_C^{-1}(P)$ consists of one point, say P_Y, by Remark 3.9. We let $\mathfrak{n}$ denote the maximal ideal sheaf of $P_Y\in Y$. Then we take a log resolution $q\colon W\to Y$ of $(Y,\mathfrak{a}\mathfrak{m}\mathscr{O}_Y\cdot\mathfrak{n})$ and set the composition $g=f\circ q\colon W\to X$. We have the following diagram.

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[dr, "g"'] & Y \arrow[d, "f"] & \supset & C_Y \arrow[d, "f_C"] & \ni & P_Y \arrow[d] \\ & X & \supset & C & \ni & P \end{tikzcd}$$

The normality of C is equivalent to the isomorphism $\mathscr{O}_C\simeq f_{C*}\mathscr{O}_{C_Y}$. We shall see this by constructing an ideal sheaf $\mathfrak{n}_a$ on C_Y which satisfies $f_{C*}\mathfrak{n}_a\subset\mathscr{O}_C$ and $\mathscr{O}_C/f_{C*}\mathfrak{n}_a\simeq f_{C*}\mathscr{O}_{C_Y}/f_{C*}\mathfrak{n}_a$.

We fix ε in (1) for $\Delta=-K_{Y/X}$, that is, $F=\mathrm{Nklt}_W(X,\mathfrak{a})=\mathrm{Nklt}_W(X,\mathfrak{a}^{1+\varepsilon})$. For the $\mathfrak{a}\langle l\rangle$ in (2), we consider a log resolution $f_l\colon Y_l\to X$ of $(X,\mathfrak{a}\langle l\rangle\mathfrak{m})$ which factors through f as $f_l=f\circ p_l$. We extend Lemma 3.6.

Lemma 3.10. *Let f and $f_l=f\circ p_l$ be as above. Then for an arbitrary ideal sheaf $\mathscr{I}$ on Y containing $p_{l*}\mathscr{O}_{Y_l}(-N_l)$, with N_l in Lemma* 3.6, *one has $R^1f_*\mathscr{I}=0$.*

Proof. By (1) for $\Delta=-K_{Y/X}$ and (2), we see $p_l(\mathrm{Supp}\,N_l)=\mathrm{Nklt}_Y(X,\mathfrak{a}\langle l\rangle)\subset q(F\cap g^{-1}(P))=C_Y\cap f^{-1}(P)=P_Y$, whence the cokernel $\mathscr{Q}$ of the natural injection $p_{l*}\mathscr{O}_{Y_l}(-N_l)\hookrightarrow\mathscr{I}$ is a skyscraper sheaf. In particular, $R^1f_*\mathscr{Q}=0$. Apply f_* to the exact sequence

$$0\to p_{l*}\mathscr{O}_{Y_l}(-N_l)\to\mathscr{I}\to\mathscr{Q}\to 0.$$

By Lemma 3.6 and $R^1f_*\mathscr{Q}=0$, we obtain $R^1f_*\mathscr{I}=0$. q.e.d.

Since C is the unique lc centre of $(X,\mathfrak{a})$, every irreducible component of F maps onto C_Y. Thus $F\cap q^{-1}(P_Y)\neq\emptyset$ and any irreducible component D of $F\cap q^{-1}(P_Y)$ has dimension 1. We fix one such D, and let $E_D\subset q^{-1}(P_Y)$ and $F_D\subset F$ be the unique prime divisors such that $D\subset E_D\cap F_D$. We derive a vanishing for ideal sheaves on Y close to that of C_Y.

Lemma 3.11. *$R^1f_*(q_*(\mathscr{O}_W(-aE_D)+\mathscr{O}_W(-F_D)))=0$ for any $a\in\mathbb{N}$.*

Proof. Take the tower of blow-ups in (3). For fixed a, choose $i\in\mathbb{N}$ such that $a_{E_D}(X,\mathfrak{a}^{1+\varepsilon})-i\varepsilon\operatorname{ord}_{F_D}\mathfrak{a}\leq -a$. Then Lemma 3.4 for $\Delta=-K_{Y/X}$ shows

(4) $\quad h_{i*}\mathscr{O}_{W_i}(\lceil a_{E_i}(X,\mathfrak{a}^{1+\varepsilon})\rceil E_i)\subset h_{i*}\mathscr{O}_{W_i}(-aE_i)\subset\mathscr{O}_W(-aE_D)+\mathscr{O}_W(-F_D).$

Take l such that $l\operatorname{ord}_{E_i}\mathfrak{m}\geq\operatorname{ord}_{E_i}\mathfrak{a}_j$ for all j. Then,

$$a_{E_i}(X,\mathfrak{a}^{1+\varepsilon})=a_{E_i}(X,\mathfrak{a}\langle l\rangle). \tag{5}$$

For this l, we take a log resolution $f_l\colon Y_l\to X$ of $(X,\mathfrak{a}\langle l\rangle\mathfrak{m})$ which factors through f, such that $c_{Y_l}(E_i)$ is a divisor. Then by (4) and (5), one can apply Lemma 3.10 to $\mathscr{I}=q_*(\mathscr{O}_W(-aE_D)+\mathscr{O}_W(-F_D))$. q.e.d.

Now we set the ideal sheaf $\mathfrak{n}_a$ on C_Y as

$$\mathfrak{n}_a:=q_*(\mathscr{O}_W(-aE_D)+\mathscr{O}_W(-F_D))\cdot\mathscr{O}_{C_Y}.$$

Lemma 3.12. *There exists a such that $f_{C*}\mathfrak{n}_a\subset\mathscr{O}_C$.*

Proof. Note that $\mathfrak{n}\mathscr{O}_{C_Y}$ is an invertible ideal sheaf on C_Y. Set $n=\operatorname{ord}_{E_D}\mathfrak{n}$, then

$$\mathfrak{n}_{nl}\subset q_*\mathscr{O}_{F_D}(-nlE_D|_{F_D})=\mathfrak{n}^l\mathscr{O}_{C_Y} \tag{6}$$

for any l. Take an f-exceptional divisor $A\geq 0$ on Y such that $-A$ is f-ample and set $\mathscr{O}_{C_Y}(-A|_{C_Y})=\mathfrak{n}^t\mathscr{O}_{C_Y}$. By Serre vanishing theorem [10, Théorème III.2.2.1], there exists m_0 such that $R^1f_*\mathscr{I}_{C_Y}(-mA)=0$ for any $m\geq m_0$, where $\mathscr{I}_{C_Y}$ is the ideal sheaf of C_Y on Y. Then we have the surjection $f_*\mathscr{O}_Y(-mA)\twoheadrightarrow f_{C*}\mathscr{O}_{C_Y}(-mA|_{C_Y})=f_{C*}\mathfrak{n}^{tm}\mathscr{O}_{C_Y}$, which provides

$$f_{C*}\mathfrak{n}^{tm}\mathscr{O}_{C_Y}=f_*\mathscr{O}_Y(-mA)\cdot\mathscr{O}_C\subset\mathscr{O}_C. \tag{7}$$

Combining (6) and (7), we obtain $f_{C*}\mathfrak{n}_{ntm}\subset f_{C*}\mathfrak{n}^{tm}\mathscr{O}_{C_Y}\subset\mathscr{O}_C$ for $m\geq m_0$. q.e.d.

Proof of the normality of C. Applying f_* to the exact sequence

$$0\to q_*(\mathscr{O}_W(-aE_D)+\mathscr{O}_W(-F_D))\to\mathscr{O}_Y\to\mathscr{O}_{C_Y}/\mathfrak{n}_a\to 0$$

and using Lemma 3.11, we obtain the surjection $\mathscr{O}_X\twoheadrightarrow f_{C*}(\mathscr{O}_{C_Y}/\mathfrak{n}_a)$. This homomorphism is factored as

$$\mathscr{O}_X\twoheadrightarrow\mathscr{O}_C/f_{C*}\mathfrak{n}_a\cap\mathscr{O}_C\hookrightarrow f_{C*}\mathscr{O}_{C_Y}/f_{C*}\mathfrak{n}_a\hookrightarrow f_{C*}(\mathscr{O}_{C_Y}/\mathfrak{n}_a),$$

so we have an isomorphism $\mathscr{O}_C/f_{C*}\mathfrak{n}_a\cap\mathscr{O}_C\simeq f_{C*}\mathscr{O}_{C_Y}/f_{C*}\mathfrak{n}_a$. For a in Lemma 3.12, it is $\mathscr{O}_C/f_{C*}\mathfrak{n}_a\simeq f_{C*}\mathscr{O}_{C_Y}/f_{C*}\mathfrak{n}_a$. Therefore $\mathscr{O}_C\simeq f_{C*}\mathscr{O}_{C_Y}$, meaning the normality of C. q.e.d.

Theorem 1.2 is established.

Remark 3.13. (i) One may prove the normality of C by using Zariski's subspace theorem [1, (10.6)]. One has an isomorphism $\mathscr{O}_C/f_{C*}\mathfrak{n}_a\cap\mathscr{O}_C\simeq f_{C*}(\mathscr{O}_{C_Y}/\mathfrak{n}_a)$ for any a. By (6), the family $\{\mathfrak{n}_a\}_a$ gives the $\mathfrak{n}\mathscr{O}_{C_Y}$-adic topology. Since the family $\{f_*\mathscr{O}_Y(-mA)\}_m$ in the proof of Lemma 3.12 gives the $\mathfrak{m}$-adic topology by Zariski's subspace theorem (cf. [13, Lemma

3]), we see from (7) that the family $\{f_{C*}\mathfrak{n}_a \cap \mathscr{O}_C\}_a$ as well as $\{f_{C*}\mathfrak{n}^a\mathscr{O}_{C_Y} \cap \mathscr{O}_C\}_a$ gives the $\mathfrak{m}\mathscr{O}_C$-adic topology. Hence $\widehat{\mathscr{O}_{C,P}} \simeq \varprojlim_a \mathscr{O}_C/f_{C*}\mathfrak{n}_a \cap \mathscr{O}_C \simeq \varprojlim_a f_{C*}(\mathscr{O}_{C_Y}/\mathfrak{n}_a) \simeq \widehat{\mathscr{O}_{C_Y,P_Y}}$ and C is normal by [10, Proposition IV.2.1.13].

(ii) The author used Zariski's subspace theorem in the proof of [14, (10)], but it derives only the inclusion $\bar{\varphi}_*\mathscr{O}_{\bar{X}}(-l_2E_Z) \subset \mathscr{I}_Z^{(l_1)}$ for the l-th symbolic power $\mathscr{I}_Z^{(l)}$ of $\mathscr{I}_Z$. In order to obtain [14, (10)], we need the equivalence of the $\mathscr{I}_Z$-adic topology and the $\mathscr{I}_Z$-symbolic topology by [36, §6 Lemma 3] (see also [27], [34]).

4. The ACC for minimal log discrepancies

In this section, we discuss the ACC for minimal log discrepancies on smooth varieties from the point of view of generic limits.

4.A. Statements.

We begin with the statement of the ACC conjecture.

Definition 4.1. We say that a subset I of $\mathbb{R}$ satisfies the *ascending chain condition* (*ACC*) (resp. the *descending chain condition* (*DCC*)) if there exist no infinite strictly increasing (resp. strictly decreasing) sequences of elements in I.

Remark 4.2. $I \subset \mathbb{R}$ is finite if and only if I satisfies both the ACC and DCC.

Definition 4.3. Let $P \in (X, \Delta = \sum_i \delta_i\Delta_i, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$ be a germ of a triple. We write $\mathrm{Coef}_P(\Delta, \mathfrak{a})$ for the set which consists of all $\delta_i > 0$ and all $r_j > 0$ with $\mathfrak{a}_j$ non-trivial at P.

Conjecture 4.4 (Shokurov [28], [30], Cascini, McKernan [24])**.** *Fix $d \in \mathbb{N}$ and subsets $I \subset (0,\infty)$ and $J \subset [0,\infty)$ both of which satisfy the DCC. Then there exist finite subsets $I_0 \subset I$ and $J_0 \subset J$ such that if $P \in (X, \Delta, \mathfrak{a})$ is a germ of a triple on a variety X of dimension d with* $\mathrm{Coef}_P(\Delta, \mathfrak{a}) \subset I$ *and* $\mathrm{mld}_P(X, \Delta, \mathfrak{a}) \in J$*, then* $\mathrm{Coef}_P(\Delta, \mathfrak{a}) \subset I_0$ *and* $\mathrm{mld}_P(X, \Delta, \mathfrak{a}) \in J_0$.

Conjecture 4.4 by Cascini and McKernan is a generalisation of the original conjecture by Shokurov, which claims only the existence of J_0. When $d = 2$, the existence of J_0 was proved by Alexeev [2]. The motivation of this conjecture stems from the reduction by Shokurov [31] that the termination of flips follows from two conjectural properties of minimal log discrepancies: the ACC and the lower semi-continuity. For the purpose of the termination of flips, one may assume I in Conjecture 4.4 to be a finite set.

We consider Conjecture 4.4 with the assumption of the smoothness of X. Then we may assume $\Delta = 0$ by absorbing Δ to $\mathfrak{a}$, since any divisor on X is a Cartier divisor.

Conjecture 4.4′. *Fix $d \in \mathbb{N}$ and subsets $I \subset (0,\infty)$ and $J \subset [0,d]$ both of which satisfy the DCC. Then there exist finite subsets $I_0 \subset I$ and $J_0 \subset J$ such that if $P \in (X, \mathfrak{a})$ is a germ of a pair on a smooth variety X of dimension d with* $\mathrm{Coef}_P\,\mathfrak{a} \subset I$ *and* $\mathrm{mld}_P(X, \mathfrak{a}) \in J$*, then* $\mathrm{Coef}_P\,\mathfrak{a} \subset I_0$ *and* $\mathrm{mld}_P(X, \mathfrak{a}) \in J_0$.

Theorem 1.3 is Conjecture 4.4′ for $d = 3$ with $J \subset (1,3]$. Conjecture 4.4′ with I finite was proved in [16].

4.B. **Reduction.** We shall reduce Conjecture 4.4′ to the stability of minimal log discrepancies in taking a generic limit of $\mathbb{R}$-ideals. We refer to Appendix A for the definition of a generic limit and the relevant notation: $R = k[[x_1,\dots,x_d]]$ with maximal ideal $\mathfrak{m}$ and $X = \operatorname{Spec} R$ with closed point P, and for a field extension K of k, $R_K = K[[x_1,\dots,x_d]]$ with maximal ideal $\mathfrak{m}_K$ and $X_K = \operatorname{Spec} R_K$ with closed point P_K.

Conjecture 4.5 ([16, Conjecture 5.7])**.** *Fix $r_1,\dots,r_e > 0$. Let $S = \{(\mathfrak{a}_{i1},\dots,\mathfrak{a}_{ie})\}_{i\in I}$ be a collection of e-tuples of ideals in $R = k[[x_1,\dots,x_d]]$, and $(\mathsf{a}_1,\dots,\mathsf{a}_e)$ the generic limit of S defined in R_K with respect to a family $\mathscr{F} = (Z_l,(\bar{\mathsf{a}}_j(l))_j,I_l,s_l,t_{l+1})_{l\ge l_0}$ of approximations of S. Set $\mathfrak{a}_i = \prod_j \mathfrak{a}_{ij}^{r_j}$ and $\mathsf{a} = \prod_j \mathsf{a}_j^{r_j}$. Then after replacing $\mathscr{F}$ with a subfamily but using the same notation,*

$$\operatorname{mld}_{P_K}(X_K,\mathsf{a}) = \operatorname{mld}_P(X,\mathfrak{a}_i)$$

for any $i \in I_l$ with $l \ge l_0$.

Conjecture 4.5 is closely related to the ideal-adic semi-continuity of minimal log discrepancies.

Conjecture 4.6 (Mustaţă, cf. [14, Conjecture 2.5])**.** *Let $P \in X = \operatorname{Spec} k[[x_1,\dots,x_d]]$ and $\mathfrak{m}$ be as above and $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ an $\mathbb{R}$-ideal on X. Then there exists an integer l such that if an $\mathbb{R}$-ideal $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$ on X satisfies $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$ for all j, then* $\operatorname{mld}_P(X,\mathfrak{a}) = \operatorname{mld}_P(X,\mathfrak{b})$.

Remark 4.7. One inequality is easy in both conjectures. One has $\operatorname{mld}_{P_K}(X_K,\mathsf{a}) \ge \operatorname{mld}_P(X,\mathfrak{a}_i)$ in Conjecture 4.5 by Lemma A.8, and $\operatorname{mld}_P(X,\mathfrak{a}) \ge \operatorname{mld}_P(X,\mathfrak{b})$ in Conjecture 4.6 by [14, Remark 2.5.3]. In particular, these conjectures hold in the case when (X_K,a) (resp. $(X,\mathfrak{a})$) is not lc.

Proposition 4.8. *Conjecture* 4.5 *implies Conjectures* 4.4′ *and* 4.6.

Proof. Firstly, we shall see Conjecture 4.4′. It was observed by Mustaţă and sketched in [14, Remark 2.5.1]. Let $\{\mathfrak{a}_i = \prod_{j=1}^{e_i} \mathfrak{a}_{ij}^{r_{ij}}\}_{i\in N}$ be an arbitrary collection of $\mathbb{R}$-ideals on $X = \operatorname{Spec} R$ such that $\mathfrak{a}_{ij}$ are non-trivial at P, $r_{ij} \in I$ and

$$m_i := \operatorname{mld}_P(X,\mathfrak{a}_i) \in J.$$

Then $\sum_{j=1}^{e_i} r_{ij} \le \operatorname{ord}_E \mathfrak{a}_i \le a_E(X) = d$ for the divisor E obtained by the blow-up of X at P, since $m_i \ge 0$. The I has the minimum, say $\iota > 0$, so $e_i \le \iota^{-1}d$. By Corollary 2.3 and Remark 4.2, it is enough to show that both the subsets $\bigcup_{i\in N} \operatorname{Coef}_P \mathfrak{a}_i$ of I and $\bigcup_{i\in N}\{m_i\}$ of J satisfy the ACC. We may replace N with a countable subset $\mathbb{N}$ on which e_i is constant, say e, such that the sequences $\{r_{ij}\}_{i\in\mathbb{N}}$ for $1 \le j \le e$ and $\{m_i\}_{i\in\mathbb{N}}$ are non-decreasing. By $r_{ij} \le d$ and $m_i \le d$, these sequences have limits

$$r_j := \lim_i r_{ij} \quad \text{and} \quad m := \lim_i m_i.$$

It suffices to prove $r_{ij} = r_j$ and $m_i = m$ for some i.

For the collection $S = \{(\mathfrak{a}_{i1},\dots,\mathfrak{a}_{ie})\}_{i\in\mathbb{N}}$ of e-tuples of ideals in R, we take a family $\mathscr{F} = (Z_l,(\bar{\mathsf{a}}_j(l))_j,I_l,s_l,t_{l+1})_{l\ge l_0}$ of approximations of S and the generic limit $(\mathsf{a}_1,\dots,\mathsf{a}_e)$ of S defined in R_K with respect to $\mathscr{F}$ as in Lemma A.8, where $E_K \in \mathscr{D}_{X_K}$ computing

$$M := \operatorname{mld}_{P_K}(X_K,\prod_j \mathsf{a}_j^{r_j})$$

is fixed. It is extended to E_l over $X\times_{\operatorname{Spec}k}Z_l$, and we have $M=\operatorname{mld}_P(X,\prod_j(\mathfrak{a}_{ij}+\mathfrak{m}^l)^{r_j})=a_{(E_l)_z}(X,\prod_j(\mathfrak{a}_{ij}+\mathfrak{m}^l)^{r_j})$ and $\operatorname{ord}_{E_K}\mathsf{a}_j=\operatorname{ord}_{(E_l)_z}(\mathfrak{a}_{ij}+\mathfrak{m}^l)<l$ for $i\in I_l$ with $z=s_l(i)$ using (iii) in Definition A.1. Hence $\operatorname{ord}_{E_K}\mathsf{a}_j=\operatorname{ord}_{(E_l)_z}\mathfrak{a}_{ij}$ and

$$(8)\qquad m_i\le a_{(E_l)_z}(X,\prod_j\mathfrak{a}_{ij}^{r_{ij}})=a_{(E_l)_z}(X,\prod_j\mathfrak{a}_{ij}^{r_j})+\sum_j(r_j-r_{ij})\operatorname{ord}_{(E_l)_z}\mathfrak{a}_{ij}$$
$$=M+\sum_j(r_j-r_{ij})\operatorname{ord}_{E_K}\mathsf{a}_j.$$

By Conjecture 4.5, $M=\operatorname{mld}_P(X,\prod_j\mathfrak{a}_{ij}^{r_j})\le m_i$ for any $i\in I_l$ after replacing $\mathscr{F}$ with a subfamily. With (8), we obtain

$$M\le m_i\le M+\sum_j(r_j-r_{ij})\operatorname{ord}_{E_K}\mathsf{a}_j.$$

The right-hand side converges to M, whence $m_i=m=M$. Then $\operatorname{mld}_P(X,\prod_j\mathfrak{a}_{ij}^{r_{ij}})=\operatorname{mld}_P(X,\prod_j\mathfrak{a}_{ij}^{r_j})$, so $r_{ij}=r_j$.

Secondly, we shall see Conjecture 4.6. Suppose the contrary. Then for every $i\in\mathbb{N}$, there exists an $\mathbb{R}$-ideal $\mathfrak{b}_i=\prod_j\mathfrak{b}_{ij}^{r_j}$ on X such that $\mathfrak{a}_j+\mathfrak{m}^i=\mathfrak{b}_{ij}+\mathfrak{m}^i$ for all j but $\operatorname{mld}_P(X,\mathfrak{a})\ne\operatorname{mld}_P(X,\mathfrak{b}_i)$. Take a family $\mathscr{F}=(Z_l,(\bar{\mathsf{b}}_j(l))_j,I_l,s_l,t_{l+1})_{l\ge l_0}$ of approximations of $S=\{(\mathfrak{b}_{ij})_j\}_{i\in\mathbb{N}}$ and the generic limit $(\mathsf{b}_j)_j$ of S defined in R_K with respect to $\mathscr{F}$. Then for $l\ge l_0$,

$$\bar{\mathsf{b}}_j(l)_zR=\mathfrak{b}_{ij}+\mathfrak{m}^l=\mathfrak{a}_j+\mathfrak{m}^l$$

for $i\in I_l$ with $z=s_l(i)$ satisfying $i\ge l$, and such z form a dense subset of Z_l. This implies $\bar{\mathsf{b}}_j(l)=((\mathfrak{a}_j+\mathfrak{m}^l)\cap\bar{R})\otimes_k\mathscr{O}_{Z_l}$, whence

$$\bar{\mathsf{b}}_j(l)_K=(\mathfrak{a}_jR_K+\mathfrak{m}_K^l)\cap\bar{R}_K.$$

Then $\mathsf{b}_j=\varprojlim_l\bar{\mathsf{b}}_j(l)_K=\mathfrak{a}_jR_K$ by Remark A.3, so $\operatorname{mld}_{P_K}(X_K,\prod_j\mathsf{b}_j^{r_j})=\operatorname{mld}_P(X,\mathfrak{a})$ by Corollary 2.3. By Conjecture 4.5, we have $\operatorname{mld}_{P_K}(X_K,\prod_j\mathsf{b}_j^{r_j})=\operatorname{mld}_P(X,\mathfrak{b}_i)$ for infinitely many i, that is, $\operatorname{mld}_P(X,\mathfrak{a})=\operatorname{mld}_P(X,\mathfrak{b}_i)$, which is absurd. q.e.d.

Remark 4.9. Proposition 4.8 has the refinement that for fixed d and $a\ge 0$,

(i) Conjecture 4.5 for d with $\operatorname{mld}_{P_K}(X_K,\mathsf{a})>a$ (resp. $\ge a$) implies Conjecture 4.4′ for d with $J\subset(a,d]$ (resp. $\subset[a,d]$), and
(ii) Conjecture 4.5 for d with $\operatorname{mld}_{P_K}(X_K,\mathsf{a})=a$ implies Conjecture 4.6 for d with $\operatorname{mld}_P(X,\mathfrak{a})=a$.

This is obvious by the above proof. Note that (8) implies $m\le M$.

Remark 4.10. Theorem A.9 gives Conjecture 4.6 in the case when $\operatorname{mld}_P(X,\mathfrak{a})=0$, and then its Corollary A.10 gives Conjecture 4.5 in the case when $\operatorname{mld}_{P_K}(X_K,\mathsf{a})=0$. The order of this logic is opposite to Proposition 4.8. We expect that an effective estimate of l in Conjecture 4.6 implies Conjecture 4.5.

Theorem A.9 is reduced to the corresponding statement [6, Theorem 1.4] on a variety by the property that the log canonical threshold for an ideal in $\widehat{\mathscr{O}_{Y,Q}}$ is approximated by those for ideals in $\mathscr{O}_{Y,Q}$. This property for the minimal log discrepancy on X is a special case of Conjecture 4.5, so we do not know how to reduce Conjecture 4.6 to its variety version. The version of Conjecture 4.6 for a germ $Q\in(Y,\Delta,\mathfrak{a})$ of a triple on a variety Y holds when (i) $(Y,\Delta,\mathfrak{a})$ is klt [14, Theorem 2.6], (ii) Y is a surface [15], or (iii) Y is toric and Q, Δ, $\mathfrak{a}$ are torus invariant [25, Theorem 1.8].

The variety version of Theorem A.9 is globalised.

Theorem 4.11. *Let* $(Y,\Delta,\mathfrak{a}=\prod_j \mathfrak{a}_j^{r_j})$ *be a triple on a variety* Y *and* Z *an irreducible closed subset of* Y*. Suppose* $\mathrm{mld}_{\eta_Z}(Y,\Delta,\mathfrak{a})=0$ *and it is computed by* $E\in\mathscr{D}_Y$*. Then there exists an open subset* Y' *of* Y *containing* η_Z *such that if an* $\mathbb{R}$*-ideal* $\mathfrak{b}=\prod_j \mathfrak{b}_j^{r_j}$ *on* Y' *satisfies* $\mathfrak{a}_j|_{Y'}+\mathfrak{p}_j=\mathfrak{b}_j+\mathfrak{p}_j$ *for all* j*, where* $\mathfrak{p}_j=\{u\in\mathscr{O}_{Y'}\mid \mathrm{ord}_E u>\mathrm{ord}_E \mathfrak{a}_j\}$*, then* $(Y',\Delta|_{Y'},\mathfrak{b})$ *is lc about* $Z|_{Y'}$ *and* $\mathrm{mld}_{\eta_Z}(Y',\Delta|_{Y'},\mathfrak{b})=0$*.*

Proof. Take a log resolution $f\colon W\to Y$ of $(Y,\Delta,\mathfrak{a}\mathscr{I}_Z)$, where $\mathscr{I}_Z$ is the ideal sheaf of Z, such that E is realised as a divisor on W. Then $F:=\mathrm{Exc}\,f\cup\mathrm{Supp}\,\Delta_W\cup\mathrm{Cosupp}\,\mathfrak{a}\mathscr{I}_Z\mathscr{O}_Y$ is an snc divisor $\sum_i F_i$, where Δ_W is the strict transform of Δ. By generic smoothness [11, Corollary III.10.7], there exists an open subset Y' of Y containing η_Z such that if the restriction $S'=S|_{f^{-1}(Y')}$ of a stratum S of $\sum_i F_i$ satisfies $S'\neq\emptyset$ and $f(S')\subset Z'=Z|_{Y'}$, then $S'\to Z'$ is smooth and surjective.

Set $z=\dim Z$. We claim that for any $Q\in Z'$,

$$\mathrm{mld}_Q(Y,\Delta,\mathfrak{a}\mathfrak{m}_Q^z)=0\quad\text{and it is computed by } G_Q \tag{9}$$

for the maximal ideal sheaf $\mathfrak{m}_Q$ and the divisor G_Q obtained by the blow-up of W along a component of $E\cap f^{-1}(Q)$. This can be verified from the local description at each closed point $R\in f^{-1}(Q)$ that there exists a regular sequence of parameters $v_1,\dots,v_{\dim Y}\in\mathscr{O}_{Y,R}$ such that $\mathfrak{m}_Q\mathscr{O}_{Y,R}=(v_1,\dots,v_z,\prod_{l=1}^s v_{z+l})\mathscr{O}_{Y,R}$ and F is given by $\prod_{l=1}^t v_{z+l}=0$ for some s,t with $1\le s\le t$.

Because $\mathrm{ord}_{G_Q}\mathfrak{a}_j=\mathrm{ord}_E\mathfrak{a}_j$ and $\mathrm{ord}_{G_Q}u\ge\mathrm{ord}_E u$ for $u\in\mathscr{O}_{Y'}$, we conclude $\mathrm{mld}_Q(Y',\Delta|_{Y'},\mathfrak{b}\mathfrak{m}_Q^z)=0$ for $\mathfrak{b}$ in Theorem 4.11 by [6, Theorem 1.4] (its proof works for triples). Hence $(Y',\Delta|_{Y'},\mathfrak{b})$ is lc about Z', and $\mathrm{mld}_{\eta_Z}(Y',\Delta|_{Y'},\mathfrak{b})=0$ by $a_E(Y',\Delta|_{Y'},\mathfrak{b})=0$. q.e.d.

Corollary 4.12. *Let* $(Y,\Delta,\mathfrak{a}=\prod_j \mathfrak{a}_j^{r_j})$ *be an lc triple on a variety* Y *and* Z *a closed subset of* Y *with ideal sheaf* $\mathscr{I}_Z$*. Then there exists an integer* l *such that if an* $\mathbb{R}$*-ideal* $\mathfrak{b}=\prod_j \mathfrak{b}_j^{r_j}$ *on* Y *satisfies* $\mathfrak{a}_j+\mathscr{I}_Z^l=\mathfrak{b}_j+\mathscr{I}_Z^l$ *for all* j*, then* $(Y,\Delta,\mathfrak{b})$ *is lc about* Z*.*

Remark 4.13. The author should have written the proof after [14, Theorem 2.4]. The estimate of l in [14, Remark 2.4.1] is incorrect unless Z is a closed point (so is that of l_1 in [14, Lemma 3.1]).

5. The klt and plt cases

In this section, we settle Conjecture 4.5 in the klt case, and in the plt case whose lc centre has an isolated singularity. Conjecture 4.5 in these cases will be applied in the proof of Theorem 1.3 in Section 6. We keep the notation in Appendix A, so $P\in X=\mathrm{Spec}\,R$ with $R=k[[x_1,\dots,x_d]]$ and $P_K\in X_K=\mathrm{Spec}\,R_K$ with $R_K=K[[x_1,\dots,x_d]]$. In the course of the proofs, we will often replace the family $\mathscr{F}$ with a subfamily, but we keep using the same notation $\mathscr{F}=(Z_l,(\bar{\mathsf{a}}_j(l))_j,I_l,s_l,t_{l+1})_{l\ge l_0}$ to avoid intricacy.

5.A. The klt case.

Theorem 5.1. *Conjecture* 4.5 *holds in the case when* (X_K,a) *is klt.*

Proof. It is shown similarly to [14, Theorem 2.6]. By Remark 4.7, it suffices to show that after replacing $\mathscr{F}$ with a subfamily,

$$a_E(X,\mathfrak{a}_i)\ge\mathrm{mld}_{P_K}(X_K,\mathsf{a}) \tag{10}$$

for any $i \in I_l$ (with $l \geq l_0$) and any $E \in \mathscr{D}_X$ with centre P.

Take a subfamily in Lemma A.8 so that $\mathrm{mld}_P(X, \prod_j (\bar{\mathsf{a}}_j(l)_z R)^{r_j}) = \mathrm{mld}_{P_K}(X_K, \mathsf{a})$ for $z \in Z_l$. Then for $i \in I_l$,

$$a_E(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j}) \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a}) \tag{11}$$

by (iii) in Definition A.1. Since (X_K, a) is klt, we can fix $t > 0$ such that (X_K, a^{1+t}) is lc. By Corollary A.10, $(X, \mathfrak{a}_i^{1+t})$ is lc for $i \in I_l$ after replacing $\mathscr{F}$ with a subfamily, whence $a_E(X, \mathfrak{a}_i) \geq t \operatorname{ord}_E \mathfrak{a}_i = t \sum_j r_j \operatorname{ord}_E \mathfrak{a}_{ij}$. We fix $l \geq l_0$ such that $l \geq (tr_j)^{-1} \mathrm{mld}_{P_K}(X_K, \mathsf{a})$ for all j. Then,

$$a_E(X, \mathfrak{a}_i) \geq l^{-1} \operatorname{ord}_E \mathfrak{a}_{ij} \cdot \mathrm{mld}_{P_K}(X_K, \mathsf{a}) \tag{12}$$

for any j and $i \in I_l$.

If $\operatorname{ord}_E \mathfrak{a}_{ij} < l$ for all j, then $\operatorname{ord}_E \mathfrak{a}_{ij} = \operatorname{ord}_E(\mathfrak{a}_{ij} + \mathfrak{m}^l)$, so one has $a_E(X, \mathfrak{a}_i) = a_E(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j})$, and (10) follows from (11). If $\operatorname{ord}_E \mathfrak{a}_{ij} \geq l$ for some j, then (10) follows from (12). q.e.d.

Remark 5.2. By Remark 4.7, Theorem 5.1 and Corollary A.10, Conjecture 4.5 remains open only when (X_K, a) is non-klt with $\mathrm{mld}_{P_K}(X_K, \mathsf{a}) > 0$.

5.B. **The plt case whose lc centre has an isolated singularity.** Suppose that (X_K, a) is an lc but not klt pair every lc centre of which has codimension 1. Then by Proposition 3.7, (X_K, a) has the smallest lc centre S_K and it is normal. We prove Conjecture 4.5 on the assumption that S_K has an isolated singularity. Our argument has its origin in [15], but is highly cumbersome.

Theorem 5.3. *Conjecture* 4.5 *holds in the case when* (X_K, a) *has the smallest lc centre of codimension* 1 *which is regular outside* P_K.

We let S_K denote the smallest lc centre of (X_K, a). S_K is a prime divisor which is regular outside P_K. We define an $\mathbb{R}$-ideal $\mathsf{c} = \prod_j \mathsf{c}_j^{r_j}$ by the expression

$$\mathsf{a}_j = \mathsf{c}_j \mathscr{O}_{X_K}(-(\operatorname{ord}_{S_K} \mathsf{a}_j) S_K)$$

so that c_j does not vanish along S_K. The a and $\mathsf{c}\mathscr{O}_{X_K}(-S_K)$ take the same order along any divisor over X_K. We can fix $t > 0$ such that $(X_K, S_K, \mathsf{c}^{1+t})$ is lc, since S_K is the unique lc centre of (X_K, S_K, c).

We take a log resolution $f_K \colon Y_K \to X_K$ of $(X_K, S_K, \mathfrak{m}_K)$, which is isomorphic outside P_K. Let $\{E_{\alpha K}\}_\alpha$ be the set of all f_K-exceptional prime divisors. The $E_K = \sum_\alpha E_{\alpha K}$ is snc. Let $(Y_K, \Delta_K, \mathsf{a}' = \prod_j (\mathsf{a}'_j)^{r_j})$ be the pull-back of $(X_K, 0, \mathsf{a})$ and $(Y_K, T_K + \Delta_K, \mathsf{c}')$ that of (X_K, S_K, c). Then T_K is the strict transform of S_K. We set

$$L_K := T_K \cap f_K^{-1}(P_K),$$
$$C_K := \operatorname{Cosupp} \mathsf{c}' \cap f_K^{-1}(P_K).$$

We have the following diagram.

$$\begin{tikzcd}
T_K \arrow[d] & \subset & Y_K \arrow[d, "f_K"'] & \supset & E_K \arrow[d] & \supset & L_K,\, C_K \\
S_K & \subset & X_K & \ni & P_K & &
\end{tikzcd}$$

By blowing up Y_K further, we may assume that C_K is contained in the union of those $E_{\alpha K}$ satisfying

$$t\,\mathrm{ord}_{E_{\alpha K}}\mathsf{c} \geq \mathrm{mld}_{P_K}(X_K,\mathsf{a}). \tag{13}$$

One can see this by induction on $\max_J\{\min_{\alpha\in J}\{\mathrm{ord}_{E_{\alpha K}}\mathsf{c}\}\}$ in which one considers all subsets J of indices satisfying $C_K \subset \bigcup_{\alpha\in J} E_{\alpha K}$. Indeed, suppose that a subset J gives the maximum of $\min_{\alpha\in J}\{\mathrm{ord}_{E_{\alpha K}}\mathsf{c}\}$. Take a log resolution $p_K\colon Y'_K \to Y_K$ of $(Y_K, T_K+E_K, \mathscr{I}_{C_K})$ for the ideal sheaf $\mathscr{I}_{C_K}$ of C_K, and write the snc divisor $\mathrm{Exc}\,p_K$ as $\sum_{\beta\in J'} E'_{\beta K}$. Then the new C'_K on Y'_K defined like C_K is contained in $\bigcup_{\beta\in J'} E'_{\beta K}$. Further for any $\beta\in J'$, there exists $\alpha\in J$ such that $p_K(E'_{\beta K}) \subset E_{\alpha K}$, for which $\mathrm{ord}_{E'_{\beta K}}\mathsf{c} > \mathrm{ord}_{E_{\alpha K}}\mathsf{c}$. Hence

$$\min_{\beta\in J'}\{\mathrm{ord}_{E'_{\beta K}}\mathsf{c}\} > \min_{\alpha\in J}\{\mathrm{ord}_{E_{\alpha K}}\mathsf{c}\},$$

so the induction can be proceeded. Note that the order of c takes value in the discrete subset $\sum_j r_j\mathbb{Z}_{\geq 0}$ of $\mathbb{R}$.

The f_K is descendible by Proposition A.7, so replacing $\mathscr{F}$ with a subfamily, we obtain the diagram (16) in which $\bar{f}_l$ is a family of log resolutions. Shrinking Z_l, we may assume that $E_{\alpha K}$, L_K and C_K are the base changes of flat families $\bar{E}_{\alpha l}$, $\bar{L}_l$ and $\bar{C}_l$ in $\bar{Y}_l$ over Z_l. We may assume that $\sum_\alpha \bar{E}_{\alpha l}$ is an snc divisor such that for every stratum $\bar{S}_l$ of $\sum_\alpha \bar{E}_{\alpha l}$, the projections $\bar{S}_l \to Z_l$ and $\bar{S}_l \cap \bar{L}_l \to Z_l$ are smooth and surjective. We may also assume that $\mathrm{ord}_{(\bar{E}_{\alpha l})_z}\bar{\mathsf{a}}_j(l)_z$ is constant on $z\in Z_l$ for each α and j. Their base changes in Y_l are denoted by $E_{\alpha l}$, L_l and C_l. We write $\bar{E}_l = \sum_\alpha \bar{E}_{\alpha l}$ and $E_l = \sum_\alpha E_{\alpha l}$.

We fix m such that $m\,\mathrm{ord}_{E_{\alpha K}}\mathfrak{m}_K \geq \mathrm{ord}_{E_{\alpha K}}\mathsf{c}_j$ for all α and j, and set

$$\mathsf{d} := \prod_j (\mathsf{c}_j + \mathfrak{m}_K^m)^{tr_j}.$$

Then $\mathrm{ord}_{E_{\alpha K}}\mathsf{d} = t\,\mathrm{ord}_{E_{\alpha K}}\mathsf{c}$, and (X_K, ad) is lc. The d is defined over some $k(Z_l)$, so by replacing $\mathscr{F}$ with a subfamily, we may assume that d is the base change of an $\mathbb{R}$-ideal $\bar{\mathfrak{d}}_l = \prod_j \bar{\mathfrak{d}}_{lj}^{tr_j}$ on $\mathbb{A}^d_k \times_{\mathrm{Spec}\,k} Z_l$ with $\bar{\mathfrak{m}}^m \otimes_k \mathscr{O}_{Z_l} \subset \bar{\mathfrak{d}}_{lj}$, and that $\mathrm{ord}_{(\bar{E}_{\alpha l})_z}(\bar{\mathfrak{d}}_l)_z$ is constant on Z_l for each α. By Corollary A.10, after taking a subfamily, $(X, \mathfrak{a}_i(\mathfrak{d}_l)_z)$ is lc for any $i\in I_l$ with $z = s_l(i)$, where $\mathfrak{d}_l$ is the pull-back on $X\times_{\mathrm{Spec}\,k} Z_l$ of $\bar{\mathfrak{d}}_l$.

We fix $l \geq l_0$ such that

$$l\,\mathrm{ord}_{E_{\alpha K}}\mathfrak{m}_K > \mathrm{ord}_{E_{\alpha K}}\mathsf{a}_j + \mathrm{ord}_{S_K}\mathsf{a}_j \tag{14}$$

for all α and j. By Remark 4.7, for Theorem 5.3 it suffices to prove that after shrinking Z_l (and taking a subfamily accordingly),

$$a_E(X, \mathfrak{a}_i) \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a}) \tag{15}$$

for any $i\in I_l$ and $E\in\mathscr{D}_X$ with centre P. Setting $z = s_l(i)$, we shall prove (15) by treating the three cases according to the position of $c_{(Y_l)_z}(E)$:

(a) $c_{(Y_l)_z}(E) \not\subset (L_l\cup C_l)_z$.
(b) $c_{(Y_l)_z}(E) \subset (C_l)_z$.
(c) $c_{(Y_l)_z}(E) \subset (L_l)_z$ and $c_{(Y_l)_z}(E) \not\subset (C_l)_z$.

We let $\bar{\mathsf{a}}'(l) = \prod_j \bar{\mathsf{a}}'_j(l)^{r_j}$ be the weak transform on $\bar{Y}_l$ of $\prod_j \bar{\mathsf{a}}_j(l)^{r_j}$, and $\mathfrak{a}'_i = \prod_j(\mathfrak{a}'_{ij})^{r_j}$ the weak transform on $(Y_l)_z$ of $\mathfrak{a}_i$.

Lemma 5.4. (i) $\bar{\mathsf{a}}'_j(l)_K\mathscr{O}_{Y_K} = \mathsf{a}'_j + \mathsf{I}_{lj}$ *with an ideal sheaf* I_{lj} *which is contained in* $\mathscr{O}_{Y_K}(-E_K)^{\mathrm{ord}_{s_K}\mathsf{a}_j+1}$.

(ii) $\bar{\mathsf{a}}'_j(l)_z\mathscr{O}_{(Y_l)_z} = \mathfrak{a}'_{ij} + \mathscr{I}_{lij}$ *with an ideal sheaf* $\mathscr{I}_{lij}$ *which is contained in* $\mathscr{O}_{(Y_l)_z}(-(E_l)_z)^{\mathrm{ord}_{s_K}\mathsf{a}_j+1}$.

(iii) $\mathrm{Cosupp}\,\bar{\mathsf{a}}'(l) = \bar{L}_l \cup \bar{C}_l$ *after shrinking* Z_l.

Proof. Write $\mathfrak{m}_K\mathscr{O}_{Y_K} = \mathscr{O}_{Y_K}(-M_K)$ and $\mathsf{a}_j\mathscr{O}_{Y_K} = \mathsf{a}'_j\mathscr{O}_{Y_K}(-A_{jK})$. The inequality (14) means that $\mathsf{I}_{lj} = \mathscr{O}_{Y_K}(A_{jK} - lM_K)$ is an ideal sheaf contained in $\mathscr{O}_{Y_K}(-E_K)^{\mathrm{ord}_{s_K}\mathsf{a}_j+1}$. Then $\mathfrak{m}^l_K\mathscr{O}_{Y_K} = \mathsf{I}_{lj}\mathscr{O}_{Y_K}(-A_{jK})$. By $\bar{\mathsf{a}}_j(l)_KR_K = \mathsf{a}_j + \mathfrak{m}^l_K$, we have

$$\bar{\mathsf{a}}_j(l)_K\mathscr{O}_{Y_K} = \mathsf{a}'_j\mathscr{O}_{Y_K}(-A_{jK}) + \mathsf{I}_{lj}\mathscr{O}_{Y_K}(-A_{jK}),$$

which induces (i). From (i),

$$\mathrm{Cosupp}(\bar{\mathsf{a}}'(l)_K\mathscr{O}_{Y_K}) = \mathrm{Cosupp}\,\mathsf{a}' \cap f_K^{-1}(P_K) = L_K \cup C_K,$$

which is extended to $\mathrm{Cosupp}\,\bar{\mathsf{a}}'(l) = \bar{L}_l \cup \bar{C}_l$ in (iii). On the other hand, $\mathrm{ord}_{E_{\alpha K}}\mathfrak{m}_K = \mathrm{ord}_{(E_{\alpha l})_z}\mathfrak{m}$ and $\mathrm{ord}_{E_{\alpha K}}\mathsf{a}_j = \mathrm{ord}_{E_{\alpha K}}\bar{\mathsf{a}}_j(l)_KR_K = \mathrm{ord}_{(\bar{E}_{\alpha l})_z}\bar{\mathsf{a}}_j(l)_z = \mathrm{ord}_{(E_{\alpha l})_z}\mathfrak{a}_{ij}$ by (14) and Definitions A.1, A.2. Then, (ii) is induced similarly to (i). q.e.d.

The cases (a) and (b) are not difficult.

Proof of (15) *in the case* (a). Set $\Delta_l = \sum_\alpha(1 - a_{E_{\alpha K}}(X_K, \mathsf{a}))E_{\alpha l}$, base-changed to Δ_K. Then $((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i)$ is the pull-back of $(X, 0, \mathfrak{a}_i)$. For a divisor $(E_{\alpha l})_z$ containing $c_{(Y_l)_z}(E)$, we have

$$\begin{aligned} a_E((Y_l)_z, (\Delta_l)_z) &\geq \mathrm{ord}_E(E_l - \Delta_l)_z \\ &\geq \mathrm{ord}_{(E_{\alpha l})_z}(E_l - \Delta_l)_z = a_{E_{\alpha K}}(X_K, \mathsf{a}) \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a}), \end{aligned}$$

where the first inequality follows from the log canonicity of $((Y_l)_z, (E_l)_z)$. By Lemma 5.4(ii) and (iii), $\mathrm{Cosupp}\,\mathfrak{a}'_i \cap (f_l)_z^{-1}(P) = \mathrm{Cosupp}\,\bar{\mathsf{a}}'(l)_z\mathscr{O}_{(Y_l)_z} = (L_l \cup C_l)_z$, so $\mathrm{ord}_E\,\mathfrak{a}'_i = 0$. Thus

$$a_E(X, \mathfrak{a}_i) = a_E((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i) = a_E((Y_l)_z, (\Delta_l)_z) \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a}).$$

q.e.d.

Proof of (15) *in the case* (b). The $c_{(Y_l)_z}(E)$ lies on some $(E_{\alpha l})_z$ such that $E_{\alpha K}$ satisfies (13). Then

$$a_E(X, \mathfrak{a}_i) \geq \mathrm{ord}_E(\mathfrak{d}_l)_z \geq \mathrm{ord}_{(E_{\alpha l})_z}(\mathfrak{d}_l)_z = \mathrm{ord}_{E_{\alpha K}}\mathsf{d} = t\,\mathrm{ord}_{E_{\alpha K}}\mathsf{c} \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a}),$$

whose first inequality follows from the log canonicity of $(X, \mathfrak{a}_i(\mathfrak{d}_l)_z)$. q.e.d.

The case (c) is reduced to the following log canonicity.

Lemma 5.5. *After shrinking* Z_l*, the triple* $((Y_l)_z, (E_l)_z, \mathfrak{a}'_i)$ *is lc about* $(L_l)_z \setminus (C_l)_z$ *for any* $i \in I_l$ *with* $z = s_l(i) \in Z_l$.

Proof of (15) *in the case* (c) *from Lemma* 5.5. For $\Delta_l = \sum_\alpha(1 - a_{E_{\alpha K}}(X_K, \mathsf{a}))E_{\alpha l}$, we have $a_E(X, \mathfrak{a}_i) = a_E((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i) \geq \mathrm{ord}_E(E_l - \Delta_l)_z$ by Lemma 5.5, and have seen $\mathrm{ord}_E(E_l - \Delta_l)_z \geq \mathrm{mld}_{P_K}(X_K, \mathsf{a})$ in the proof in the case (a). q.e.d.

Proof of Lemma 5.5. Pick any open stratum F_K of the snc divisor E_K, which is extended to an open stratum $\bar{F}_l$ of $\bar{E}_l$. We prove Lemma 5.5 by noetherian induction. Recall that l has been fixed. Let $\bar{Q}_l$ be an irreducible locally closed subset of $\bar{F}_l \cap \bar{L}_l \setminus \bar{C}_l$ which dominates Z_l. It suffices to show the existence of a dense open subset $\bar{Q}^\circ_l$ of $\bar{Q}_l$ such that the triple $((Y_l)_z, (E_l)_z, \mathfrak{a}'_i)$ is lc about $(Q^\circ_l)_z$ for $i \in I_l$ with

$z = s_l(i)$, where $Q_l^\circ = \bar{Q}_l^\circ \times_{\bar{Y}_l} Y_l$. Indeed, start with a component $\bar{Q}_l$ of $\bar{F}_l \cap \bar{L}_l \setminus \bar{C}_l$ and find $\bar{Q}_l^\circ$. Take a dense open subset Z_l' of Z_l such that each irreducible component $\bar{Q}_l'$ of $\bar{Q}_l \setminus \bar{Q}_l^\circ|_{Z_l'}$ dominates Z_l'. Replace Z_l with Z_l' and continue the argument for each $\bar{Q}_l'$. Eventually we attain Z_l such that $((Y_l)_z, (E_l)_z, \mathfrak{a}_i')$ is lc about $(F_l \cap L_l \setminus C_l)_z$ for $i \in I_l$ with $z = s_l(i) \in Z_l$. Applying this to all open strata of E_K, we obtain a shrunk Z_l in Lemma 5.5.

We shall construct $\bar{Q}_l^\circ$. By shrinking $\bar{Q}_l$ and Z_l, we may assume that $\bar{Q}_l \to Z_l$ is smooth and surjective. Let $\bar{f}_l^+ \colon \bar{Y}_l^+ \to \mathbb{A}_k^d \times_{\operatorname{Spec} k} \bar{Q}_l$ be the base change of $\bar{f}_l$ by $\bar{Q}_l \to Z_l$. Then $\mathrm{pr}_{\bar{Q}_l} \circ \bar{f}_l^+$ has the natural section $\bar{g}_l \colon \bar{Q}_l \to \bar{Y}_l^+ = \bar{Y}_l \times_{Z_l} \bar{Q}_l$ by the immersion $\bar{Q}_l \hookrightarrow \bar{Y}_l$. We construct f_K^+ and g_K similarly for $Q_K = \bar{Q}_l \times_{\bar{Y}_l} Y_K$ as below.

$$\begin{tikzcd}
Y_K^+ \arrow[r] \arrow[d, "f_K^+"'] & \bar{Y}_l^+ \arrow[r] \arrow[d, "\bar{f}_l^+"'] & \bar{Y}_l \arrow[d, "\bar{f}_l"'] \\
X_K \times_{\operatorname{Spec} K} Q_K \arrow[r] \arrow[d] & \mathbb{A}_k^d \times_{\operatorname{Spec} k} \bar{Q}_l \arrow[r] \arrow[d] & \mathbb{A}_k^d \times_{\operatorname{Spec} k} Z_l \arrow[d] \\
Q_K \arrow[r] \arrow[uu, bend left=60, "g_K"] & \bar{Q}_l \arrow[r] \arrow[uu, bend left=60, "\bar{g}_l"] & Z_l
\end{tikzcd}$$

The $\bar{Y}_l^+$, Y_K^+ are the base changes of $\bar{Y}_l$, Y_K by smooth morphisms. For a scheme or a sheaf $\square$ on $\bar{Y}_l$ or Y_K, we mean by $\square^+$ the base change of $\square$ on $\bar{Y}_l^+$ or Y_K^+. For example, $\mathsf{a}'^+ = \prod_j (\mathsf{a}_j'^+)^{r_j} = \mathsf{a}'\mathscr{O}_{Y_K^+}$. Let $\bar{\mathfrak{q}}_l$ be the ideal sheaf of $\bar{g}_l(\bar{Q}_l)$ on $\bar{Y}_l^+$ and $\bar{G}_l \in \mathscr{D}_{\bar{Y}_l^+}$ the divisor obtained by the blow-up of $\bar{Y}_l^+$ along $\bar{\mathfrak{q}}_l$. They are base-changed to $\mathfrak{q}$ on Y_K^+ and $G_K \in \mathscr{D}_{Y_K^+}$.

Set $n = \dim F_K - 1$. Similarly to (9), we see that $\mathrm{mld}_{\eta_{g_K(Q_K)}}(Y_K^+, E_K^+, \mathfrak{q}^n \mathsf{a}'^+) = 0$ and it is computed by G_K. We have $\bar{\mathsf{a}}'(l)_K^+ \mathscr{O}_{Y_K^+} = \prod_j (\mathsf{a}_j'^+ + \mathsf{l}_{lj}^+)^{r_j}$ and $\mathrm{ord}_{G_K} \mathsf{a}_j'^+ = \mathrm{ord}_{S_K} \mathsf{a}_j < \mathrm{ord}_{G_K} \mathsf{l}_{lj}^+$ from Lemma 5.4(i), so

$$\mathrm{mld}_{\eta_{g_K(Q_K)}}(Y_K^+, E_K^+, \mathfrak{q}^n \bar{\mathsf{a}}'(l)_K^+ \mathscr{O}_{Y_K^+}) = 0 \quad \text{and it is computed by } G_K.$$

Then $\mathrm{mld}_{\eta_{\bar{g}_l(\bar{Q}_l)}}(\bar{Y}_l^+, \bar{E}_l^+, \bar{\mathfrak{q}}_l^n \bar{\mathsf{a}}'(l)^+) = 0$ and it is computed by $\bar{G}_l$. We regard $\bar{Y}_l^+$ as a family over $\bar{Q}_l$. There exists a dense open subset $\bar{Q}_l^\circ$ of $\bar{Q}_l$ such that for any closed point $q \in Q_l^\circ = \bar{Q}_l^\circ \times_{\bar{Y}_l} Y_l$ with its image $z \in Z_l$,

$$\mathrm{mld}_q((Y_l)_z, (E_l)_z, \mathfrak{m}_q^n \bar{\mathsf{a}}'(l) \mathscr{O}_{(Y_l)_z}) = 0 \quad \text{and it is computed by } (G_l)_q,$$

and $\mathrm{ord}_{(G_l)_q} \bar{\mathsf{a}}_j'(l) \mathscr{O}_{(Y_l)_z} = \mathrm{ord}_{S_K} \mathsf{a}_j$, where $\mathfrak{m}_q$ is the maximal ideal sheaf of $q \in (Y_l)_z$ and $G_l = \bar{G}_l \times_{\bar{Y}_l} Y_l$. The $(G_l)_q$ is obtained by the blow-up of $(Y_l)_z$ at q. If $z = s_l(i)$ with $i \in I_l$, then $\bar{\mathsf{a}}'(l)\mathscr{O}_{(Y_l)_z} = \prod_j (\mathfrak{a}_{ij}' + \mathscr{I}_{lij})^{r_j}$ and $\mathrm{ord}_{(G_l)_q} \bar{\mathsf{a}}_j'(l) \mathscr{O}_{(Y_l)_z} < \mathrm{ord}_{(G_l)_q} \mathscr{I}_{lij}$ by Lemma 5.4(ii). Applying Theorem A.9, we have $\mathrm{mld}_q((Y_l)_z, (E_l)_z, \mathfrak{m}_q^n \mathfrak{a}_i') = 0$, and the log canonicity of $((Y_l)_z, (E_l)_z, \mathfrak{a}_i')$ about $(Q_l^\circ)_z$ is concluded. q.e.d.

Theorem 5.3 is completed.

6. The threefold case

We shall prove Theorem 1.3. By Remark 4.9, the theorem follows from Conjecture 4.5 for $d = 3$ with $\mathrm{mld}_{P_K}(X_K, \mathsf{a}) > 1$. In Remark 5.2, Conjecture 4.5 is reduced to the case when (X_K,a) is an lc pair which has a minimal lc centre Z of positive dimension. If $d=3$, then by Theorem 1.2, Z is the smallest lc centre and it is normal. If $\dim Z=2$, then one can apply Theorem 5.3. If $\dim Z=1$, then $\mathrm{mld}_{P_K}(X_K,\mathsf{a})\le 1$ by Proposition 6.1. Therefore, we obtain Theorem 1.3.

Proposition 6.1. *Let $P\in(X,\mathfrak{a})$ be a germ of an lc pair on a regular R-variety X of dimension* 3 *with $R=K[[x_1,\dots,x_d]]$ whose smallest lc centre is of dimension* 1. *Then* $\mathrm{mld}_P(X,\mathfrak{a})\le 1$.

Proof. The smallest lc centre C of $(X,\mathfrak{a})$ is regular by Theorem 1.2. Setting $(X_0,\Delta_0,\mathfrak{a}_0):=(X,0,\mathfrak{a})$ and $C_0:=C$, we build a tower of finitely many blow-ups

$$X_n\to\cdots\to X_i\xrightarrow{f_i}X_{i-1}\to\cdots\to X_0=X$$

such that

(i) $f_i\colon X_i\to X_{i-1}$ is the blow-up along C_{i-1},
(ii) E_i is the exceptional divisor of f_i,
(iii) $(X_i,\Delta_i,\mathfrak{a}_i)$ is the pull-back of $(X_{i-1},\Delta_{i-1},\mathfrak{a}_{i-1})$,
(iv) C_i is a regular non-klt centre on X_i of $(X,\mathfrak{a})$ mapped onto C_{i-1}, and
(v) $a_{E_i}(X,\mathfrak{a})>0$ for $i<n$ and $a_{E_n}(X,\mathfrak{a})=0$.

Here one can prove the effectiveness $\Delta_i\ge 0$ and the regularity of C_i by induction. Indeed, if they hold for $i-1$, then $\mathrm{ord}_{E_i}\Delta_i=\mathrm{ord}_{C_{i-1}}\Delta_{i-1}+\mathrm{ord}_{C_{i-1}}\mathfrak{a}_{i-1}-1>0$ by Lemma 6.2. Unless $a_{E_i}(X,\mathfrak{a})=0$, an arbitrary lc centre C_i of $(X_i,\Delta_i,\mathfrak{a}_i)$ mapped onto C_{i-1} is of dimension 1 and is minimal. The regularity of C_i follows from Theorem 1.2.

Let F be the divisor obtained by the blow-up of X_n along an irreducible component of $E_n\cap(f_1\circ\cdots\circ f_n)^{-1}(P)$. Then $a_F(X,\mathfrak{a})=a_F(X_n,\Delta_n,\mathfrak{a}_n)\le a_F(X_n,E_n)=1$ by $\Delta_n\ge(1-a_{E_n}(X,\mathfrak{a}))E_n=E_n$. q.e.d.

Lemma 6.2. *Let $(X,\mathfrak{a})$ be a pair on a regular R-variety X and Z a non-klt centre of $(X,\mathfrak{a})$. Then* $\mathrm{ord}_Z\mathfrak{a}\ge 1$. *If in addition* $\mathrm{codim}_X Z\ge 2$, *then* $\mathrm{ord}_Z\mathfrak{a}>1$.

Proof. The lemma is obvious if Z is a divisor, so we may assume $\mathrm{codim}_X Z\ge 2$. Setting $X_0:=X$, $Z_0:=Z$ and $\mathfrak{a}_0:=\mathfrak{a}$, we build a tower of finitely many blow-ups

$$X_n\to\cdots\to X_i\xrightarrow{f_i}X_{i-1}\to\cdots\to X_0=X$$

such that

(i) f_i is the composition $X_i\xrightarrow{h_i}X'_{i-1}\xrightarrow{g_{i-1}}X_{i-1}$ of the blow-up $h_i\colon X_i\to X'_{i-1}$ along the strict transform on X'_{i-1} of Z_{i-1} and an embedded resolution $g_{i-1}\colon X'_{i-1}\to X_{i-1}$ of singularities of Z_{i-1}, in which g_{i-1} is isomorphic on the regular locus of Z_{i-1},
(ii) E_i is the exceptional divisor of h_i,
(iii) $\mathfrak{a}_i$ is the weak transform on X_i of $\mathfrak{a}_{i-1}$,
(iv) Z_i is a non-klt centre on X_i of $(X,\mathfrak{a})$ mapped onto Z_{i-1}, and
(v) $a_{E_i}(X,\mathfrak{a})>0$ for $i<n$ and $a_{E_n}(X,\mathfrak{a})\le 0$.

Supposing $\mathrm{ord}_Z\mathfrak{a}\le 1$, we shall derive by induction two inequalities

$$\mathrm{ord}_{Z_i}\mathfrak{a}_i\le 1\quad\text{and}\quad a_{E_n}(X_i,\mathfrak{a}_i)\le 0$$

for any i. The claim for $i=0$ is trivial. If they hold for $i-1$, then $\mathrm{ord}_{Z_i}\mathfrak{a}_i\le\mathrm{ord}_{V_i}\mathfrak{a}_i\le\mathrm{ord}_{Z_{i-1}}\mathfrak{a}_{i-1}\le 1$ by [12, Lemmata III.7, III.8] for an irreducible closed subset V_i of Z_i meeting the regular locus of Z_i such that $V_i\to Z_{i-1}$ is finite and surjective. Note that the symbol $v^{(1)}$ in [12] stands for the order. The $(X_{i-1}, 0, \mathfrak{a}_{i-1})$ is pulled back to $(X_i, \Delta_i, \mathfrak{a}_i)$ with $\mathrm{ord}_{E_i} \Delta_i = 1 + \mathrm{ord}_{Z_{i-1}} \mathfrak{a}_{i-1} - \mathrm{codim}_{X_{i-1}} Z_{i-1} \leq 0$, so $a_{E_n}(X_i, \mathfrak{a}_i) \leq a_{E_n}(X_i, \Delta_i, \mathfrak{a}_i) = a_{E_n}(X_{i-1}, \mathfrak{a}_{i-1}) \leq 0$.

We obtained $a_{E_n}(X_n, \mathfrak{a}_n) \leq 0$. However, it contradicts $a_{E_n}(X_n) = 1$ and $\mathrm{ord}_{E_n} \mathfrak{a}_n = 0$. q.e.d.

Appendix A. Generic limits

The generic limit is a limit of ideals. It was constructed first by de Fernex and Mustaţă [8] using ultraproducts, and then by Kollár [20] using Hilbert schemes. We set $\bar{R} = k[x_1, \dots, x_d]$ with maximal ideal $\bar{\mathfrak{m}}$, and $\mathbb{A}^d_k = \operatorname{Spec} \bar{R}$ with origin $\bar{P}$. We also set $R = k[[x_1, \dots, x_d]]$ with $\mathfrak{m} = \bar{\mathfrak{m}} R$, and $X = \operatorname{Spec} R$ with closed point P. Mostly we discuss on the spectrum of a noetherian ring, where an ideal in the ring is identified with its coherent ideal sheaf.

We introduce the notion of a family of approximated ideals by which a generic limit is defined.

Definition A.1. Let $S = \{(\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ie})\}_{i \in I}$ be a collection of e-tuples of ideals in R, indexed by an infinite set I. A *family $\mathscr{F}$ of approximations* of S consists of, with l_0 fixed, for each $l \geq l_0$,

(a) a variety Z_l,
(b) an ideal sheaf $\bar{\mathsf{a}}_j(l)$ on $\mathbb{A}^d_k \times_{\operatorname{Spec} k} Z_l$ containing $\bar{\mathfrak{m}}^l \otimes_k \mathscr{O}_{Z_l}$ for $1 \leq j \leq e$,
(c) an infinite subset I_l of I and a map $s_l \colon I_l \to Z_l(k)$, where $Z_l(k)$ is the set of k-points on Z_l, and
(d) a dominant morphism $t_{l+1} \colon Z_{l+1} \to Z_l$,

such that

(i) $\bar{\mathsf{a}}_j(l)$ gives a flat family of closed subschemes of $\mathbb{A}^d_k$ parametrised by Z_l,
(ii) the pull-back of $\bar{\mathsf{a}}_j(l)$ by $\mathrm{id}_{\mathbb{A}^d_k} \times t_{l+1}$ is $\bar{\mathsf{a}}_j(l+1) + \bar{\mathfrak{m}}^l \otimes_k \mathscr{O}_{Z_{l+1}}$,
(iii) $\mathfrak{a}_{ij} + \mathfrak{m}^l = \bar{\mathsf{a}}_j(l)_{s_l(i)} R$ for $i \in I_l$, where $\bar{\mathsf{a}}_j(l)_z$ is the ideal in $\bar{R}$ given by $\bar{\mathsf{a}}_j(l)$ at $z \in Z_l$,
(iv) $s_l(I_l)$ is dense in Z_l, and
(v) $I_{l+1} \subset I_l$ and $t_{l+1} \circ s_{l+1} = s_l|_{I_{l+1}}$.

The construction of $\mathscr{F}$ using Hilbert schemes is exposed in [6, Section 4]. In general, there exist essentially different families of approximations.

For a field extension K of k, we set $\bar{R}_K = \bar{R} \otimes_k K = K[x_1, \dots, x_d]$ with $\bar{\mathfrak{m}}_K = \bar{\mathfrak{m}} \bar{R}_K$, and $\mathbb{A}^d_K = \operatorname{Spec} \bar{R}_K$ with origin $\bar{P}_K$. We also set $R_K = \widehat{R \otimes_k K} = K[[x_1, \dots, x_d]]$ with $\mathfrak{m}_K = \mathfrak{m} R_K$, and $X_K = \operatorname{Spec} R_K$ with closed point P_K.

Definition A.2. Suppose that a family $\mathscr{F}$ of approximations of S is given as in Definition A.1. For this $\mathscr{F}$, take the union $K = \varinjlim_l K(Z_l)$ of the function fields $K(Z_l)$ of Z_l by the inclusions $t^*_{l+1} \colon K(Z_l) \hookrightarrow K(Z_{l+1})$. Then the *generic limit* of S with respect to $\mathscr{F}$ is the e-tuple $(\mathsf{a}_1, \dots, \mathsf{a}_e)$ of ideals in R_K such that

$$\mathsf{a}_j + \mathfrak{m}^l_K = \bar{\mathsf{a}}_j(l)_K R_K$$

for all $l \geq l_0$, where $\bar{\mathsf{a}}_j(l)_K$ is the ideal in $\bar{R}_K$ given by $\bar{\mathsf{a}}_j(l)$ at the natural K-point $\operatorname{Spec} K \to Z_l$.

Remark A.3. We have $\mathsf{a}_j = \varprojlim_l \bar{\mathsf{a}}_j(l)_K$, by $\bar{\mathsf{a}}_j(l)_K = \bar{\mathsf{a}}_j(l+1)_K + \bar{\mathfrak{m}}^l_K$ from (ii) in Definition A.1.

Definition A.4. Let $\mathscr{F} = (Z_l, (\bar{\mathsf{a}}_j(l))_j, I_l, s_l, t_{l+1})_{l \ge l_0}$ and $\mathscr{F}' = (Z'_l, (\bar{\mathsf{a}}'_j(l))_j, I'_l, s'_l, t'_{l+1})_{l \ge l'_0}$ be families of approximations of S. A *morphism* $\mathscr{F}' \to \mathscr{F}$ consists of dominant morphisms $f_l \colon Z'_l \to Z_l$ for $l \ge l'_0$, with $l'_0 \ge l_0$ imposed, such that

(i) $t_{l+1} \circ f_{l+1} = f_l \circ t'_{l+1}$,
(ii) the pull-back of $\bar{\mathsf{a}}_j(l)$ by $\mathrm{id}_{\mathbb{A}^d_k} \times f_l$ is $\bar{\mathsf{a}}'_j(l)$, and
(iii) $I'_l \subset I_l$ and $f_l \circ s'_l = s_l|_{I'_l}$.

An $\mathscr{F}'$ is called a *subfamily* of $\mathscr{F}$ if it is equipped with a morphism $\mathscr{F}' \to \mathscr{F}$ as above such that all f_l are open immersions.

We want to compare minimal log discrepancies over X and X_K. The comparison of those for approximated ideals is a consequence of the existence of a family of log resolutions on an open subfamily of triples and Corollary 2.3.

Lemma A.5 (cf. [16, Proposition 3.2(ii)])**.** *Notation as above. Let* $(\mathsf{a}_1, \dots, \mathsf{a}_e)$ *be the generic limit of S with respect to $\mathscr{F}$. Then after replacing $\mathscr{F}$ with a subfamily but using the same notation,*

$$\mathrm{mld}_{P_K}(X_K, \prod_j (\mathsf{a}_j + \mathfrak{m}_K^l)^{r_j}) = \mathrm{mld}_{\bar{P}}(\mathbb{A}^d_k, \prod_j \bar{\mathsf{a}}_j(l)_z^{r_j})$$

for all $r_1, \dots, r_e > 0$ and all $z \in Z_l$ with $l \ge l_0$.

We utilise a projective morphism which is descended to $\mathbb{A}^d_K$.

Definition A.6. A projective morphism $f_K \colon Y_K \to X_K$ is said to be *descendible* if there exists a projective morphism $\bar{f}_K \colon \bar{Y}_K \to \mathbb{A}^d_K$ whose base change to X_K is f_K.

Proposition A.7. *Let $f_K \colon Y_K \to X_K$ be a projective morphism of R_K-varieties which is isomorphic outside P_K. Then f_K is descendible.*

Proof. Assuming $d \ge 1$, f_K is the blow-up along an ideal $\mathfrak{n}_K$ in R_K [23, Theorem 8.1.24]. We may assume $\mathrm{codim}_{X_K} \mathrm{Cosupp}\, \mathfrak{n}_K \ge 2$, then $\mathrm{Cosupp}\, \mathfrak{n}_K \subset P_K$, that is, $\mathfrak{n}_K$ is an $\mathfrak{m}_K$-primary ideal. Thus, $\mathfrak{n}_K$ is the pull-back of the ideal $\bar{\mathfrak{n}}_K = \mathfrak{n}_K \cap \bar{R}_K$ in $\bar{R}_K$. Since blowing-up commutes with flat base change [23, Proposition 8.1.12(c)], the blow-up of $\mathbb{A}^d_K$ along $\bar{\mathfrak{n}}_K$ is base-changed to f_K. q.e.d.

Let $f_K \colon Y_K \to X_K$ be a descendible projective morphism, descended to $\bar{f}_K \colon \bar{Y}_K \to \mathbb{A}^d_K$. This $\bar{f}_K$ is defined over $k(Z_{l'_0})$ for some $l'_0 \ge l_0$. For $l \ge l'_0$, one can construct inductively a projective morphism $\bar{f}'_l \colon \bar{Y}'_l \to \mathbb{A}^d_k \times_{\operatorname{Spec} k} Z'_l$ with a smooth open subvariety Z'_l of Z_l such that (i) $\bar{Y}'_l$ is flat over Z'_l, (ii) $Z'_{l+1} \subset t_{l+1}^{-1}(Z'_l)$, and (iii) $\bar{f}'_{l+1}$ and $\bar{f}_K$ are the base changes of $\bar{f}'_l$, by generic flatness [10, Corollaire IV.11.1.5]. These Z'_l with $I'_l = s_l^{-1}(Z'_l(k))$ form a subfamily $\mathscr{F}'$ of $\mathscr{F}$. Replacing $\mathscr{F}$ with $\mathscr{F}'$, we obtain a commutative diagram

(16)
$$\begin{tikzcd}
Y_K \arrow[rr] \arrow[dr] \arrow[dd, "f_K"'] & & Y_l \arrow[dr] \arrow[dd, "f_l" near start] & \\
& \bar{Y}_K \arrow[rr] \arrow[dd, "\bar{f}_K" near start] & & \bar{Y}_l \arrow[dd, "\bar{f}_l"] \\
X_K \arrow[rr] \arrow[dr] & & X \times_{\operatorname{Spec} k} Z_l \arrow[dr] & \\
& \mathbb{A}^d_K \arrow[rr] & & \mathbb{A}^d_k \times_{\operatorname{Spec} k} Z_l
\end{tikzcd}$$

for $l \ge l_0$ (the l_0 is replaced) such that (i) Z_l is smooth, (ii) $\bar{f}_l$ is projective, (iii) $\bar{Y}_l$ is flat over Z_l, and (iv) $\bar{f}_{l+1}$, $\bar{f}_K$, f_l and f_K are the base changes of $\bar{f}_l$. In general, $X_K \to X \times_{\operatorname{Spec} k} Z_l$ is not the base change of $\mathbb{A}^d_K \to \mathbb{A}^d_k \times_{\operatorname{Spec} k} Z_l$.

Whenever an algebraic object over X_K descendible to $\mathbb{A}^d_K$ is specified, by taking a subfamily, one can construct (16) so that it comes from a flat family over Z_l. For example, suppose that $E_K \in \mathscr{D}_{X_K}$ with centre P_K is given. It is realised as a divisor on Y_K equipped with a log resolution $f_K\colon Y_K \to X_K$ of $(X_K, \mathfrak{m}_K)$, which is isomorphic outside P_K. This f_K is descended to a log resolution $\bar{f}_K$ by Proposition A.7, and $\bar{f}_K$ is extended to a family $\bar{f}_l$ of log resolutions in (16) by generic smoothness. There exists a prime divisor $\bar{E}_l$ on $\bar{Y}_l$ which is base-changed to E_K. By this observation, Lemma A.5 is refined as follows.

Lemma A.8 (cf. [16, Proposition 3.2(iii)]). *Notation as above. Fix $r_1, \dots, r_e > 0$ and $E_K \in \mathscr{D}_{X_K}$ computing $\operatorname{mld}_{P_K}(X_K, \prod_j \mathsf{a}_j^{r_j})$. Then after replacing $\mathscr{F}$ with a subfamily but using the same notation, there exists a divisor $\bar{E}_l$ over $\mathbb{A}^d_k \times_{\operatorname{Spec} k} Z_l$ for any $l \ge l_0$, base-changed to E_K, such that*

$$\operatorname{mld}_{P_K}(X_K, \prod_j \mathsf{a}_j^{r_j}) = \operatorname{mld}_{\bar{P}}(\mathbb{A}^d_k, \prod_j \bar{\mathsf{a}}_j(l)_z^{r_j}) = a_{(\bar{E}_l)_z}(\mathbb{A}^d_k, \prod_j \bar{\mathsf{a}}_j(l)_z^{r_j}),$$

$$\operatorname{ord}_{E_K} \mathsf{a}_j = \operatorname{ord}_{E_K}(\mathsf{a}_j + \mathfrak{m}_K^l) = \operatorname{ord}_{(\bar{E}_l)_z} \bar{\mathsf{a}}_j(l)_z < l,$$

for all $z \in Z_l$.

We apply the ideal-adic semi-continuity of log canonicity by Kollár, and de Fernex, Ein and Mustaţă.

Theorem A.9 ([20], [6], [7, Proposition 2.20]). *Let $Q \in Y$ be a germ of an lc variety and set $\hat{Y} = \operatorname{Spec} \widehat{\mathscr{O}_{Y,Q}}$ with closed point $\hat{Q}$. Let $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ be an $\mathbb{R}$-ideal on $\hat{Y}$. Suppose $\operatorname{mld}_{\hat{Q}}(\hat{Y}, \mathfrak{a}) = 0$ and it is computed by $\hat{E} \in \mathscr{D}_{\hat{Y}}$. If an $\mathbb{R}$-ideal $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$ on $\hat{Y}$ satisfies $\mathfrak{a}_j + \mathfrak{p}_j = \mathfrak{b}_j + \mathfrak{p}_j$ for all j, where $\mathfrak{p}_j = \{u \in \mathscr{O}_{\hat{Y}} \mid \operatorname{ord}_{\hat{E}} u > \operatorname{ord}_{\hat{E}} \mathfrak{a}_j\}$, then $\operatorname{mld}_{\hat{Q}}(\hat{Y}, \mathfrak{b}) = 0$.*

Corollary A.10. *In Lemma* A.8, *if* $\operatorname{mld}_{P_K}(X_K, \prod_j \mathsf{a}_j^{r_j}) = 0$, *then* $\operatorname{mld}_P(X, \prod_j \mathfrak{a}_{ij}^{r_j}) = 0$ *for any $i \in I_l$ on a subfamily. In particular, if $(X_K, \prod_j \mathsf{a}_j^{r_j})$ is lc, then so is $(X, \prod_j \mathfrak{a}_{ij}^{r_j})$.*

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `masayuki@kurims.kyoto-u.ac.jp`